LECTURE 16

Note: $f(x) = |x|$ is in $C^0(\mathbb{R})$, not in $C^1(\mathbb{R})$.
Using f^{n}s o.t.f. $x \mapsto x^{3/2}$, $x \mapsto x^{5/2}$ etc.,
you can show $C^m(\Omega) \subsetneq C^{m+1}(\Omega)$.

Consider $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$

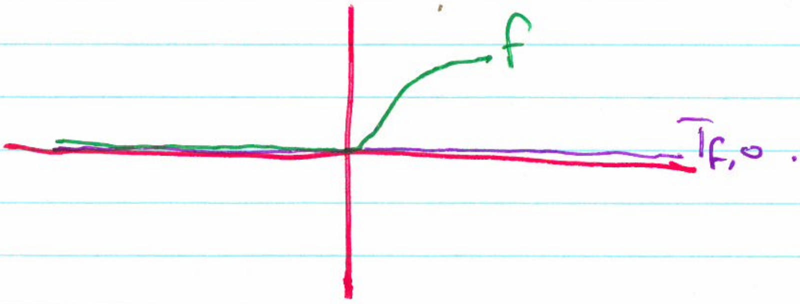
Check: * $f^{(n)}(x)$ exists $\forall x \neq 0$.

nth derivative of f at x.

* $f^{(n)}(0) = 0 \quad \forall n$.
* $f^{(n)}$ is cts on $\mathbb{R}$.

$\Rightarrow$ Taylor series for f at 0 is

$$T_{f,0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0 .$$



f is not equal to $T_{f,0}$ on any nbhd of 0.
$f \in C^\infty(\Omega)$, $f \notin C^\omega(\Omega)$.

$$C^\omega \subsetneq C^\infty \subsetneq \cdots C^{1337} \subsetneq C^{1336} \subsetneq \cdots \subsetneq C^1 \subsetneq C^0 .$$

§41–43 (8Ed §37–39) Integration.

Consider a $\mathbb{C}$-valued f^n of a real variable
$w(t) = u(t) + iv(t)$
$\in\mathbb{R}$
Define $w'(t) = u'(t) + iv'(t)$.

Standard differentiation laws for f^{n}s of a $\mathbb{R}$-valued variable apply:

* $(cw)' = cw'$ $\quad c\in\mathbb{C}$.
* $(w_1 \pm w_2)' = w_1' \pm w_2'$
* $\frac{d}{dt}(e^{ct}) = ce^{ct}$ etc.
* product, quotient rules etc.

Definite & indefinite integrals of such f^{n}s:

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \quad (0) \quad a,b\in\mathbb{R}.$$

$$\Rightarrow \mathrm{Re}\left(\int_a^b w(t)\,dt\right) = \int_a^b \mathrm{Re}(w(t))\,dt \quad (1)$$

$$\& \ \mathrm{Im}\left(\int_a^b w(t)\,dt\right) = \int_a^b \mathrm{Im}(w(t))\,dt. \quad (2)$$

$\int_a^\infty w$ etc. defined analogously.

(0) certainly makes sense for cts w, i.e., $w \in C^0([a,b])$.

Indeed, ok for so-called piecewise cts f$^{\underline{n}}$s on $[a,b]$, i.e.,

* $u, v : [a,b] \to \mathbb{R}$:

$\exists\ c_1 < c_2 < \cdots < c_n \in (a,b)$ s.t. :

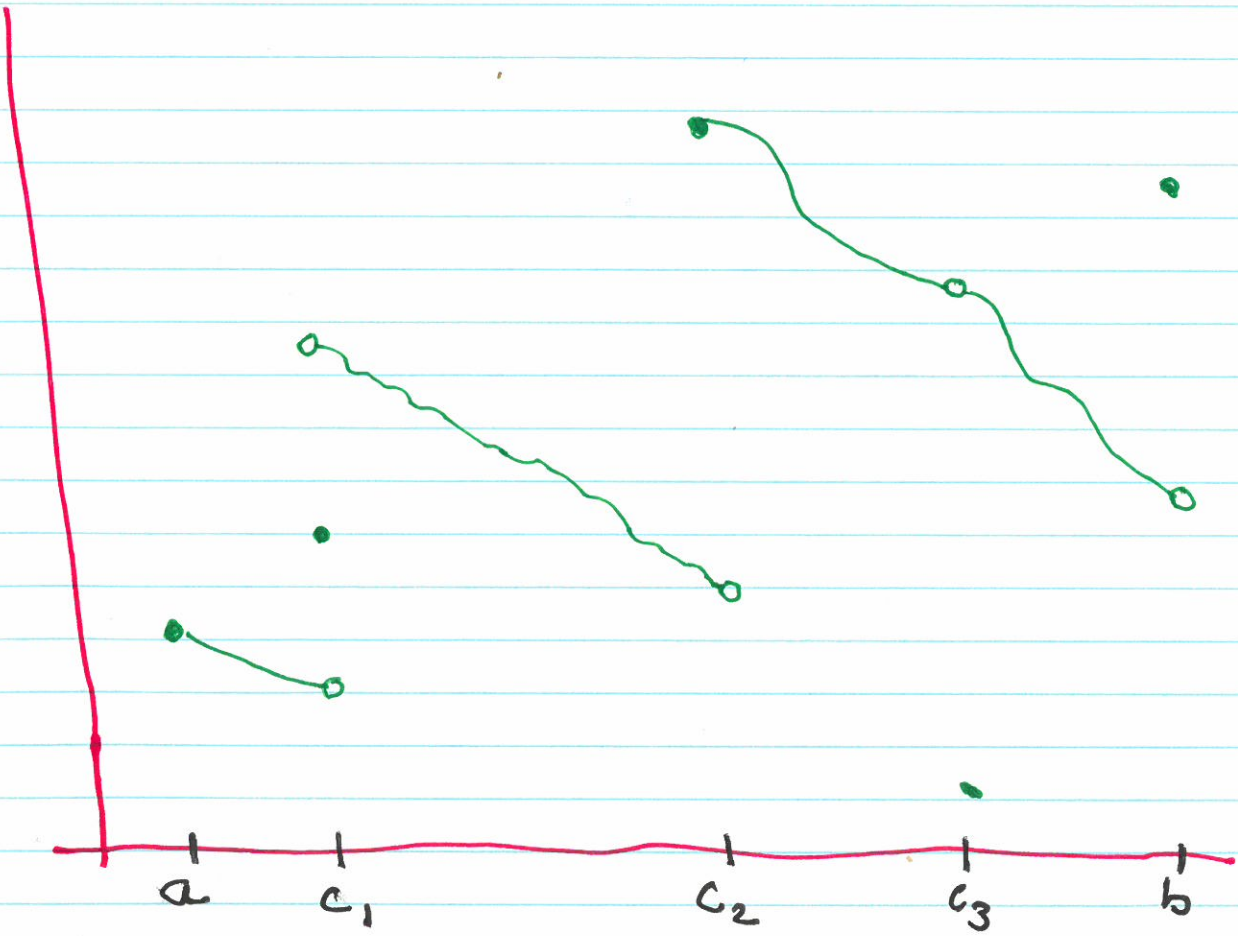
(i) w is cts on $(a,c_1), (c_1,c_2), \cdots, (c_{n-1},c_n), (c_n,b)$;

(ii) $\lim\limits_{t \to c_j^-} w(t)$, $\lim\limits_{t \to c_j^+} w(t)$ both exist (they may or may not coincide);

(iii) $\lim\limits_{t \to a^+} w(t)$, $\lim\limits_{t \to b^-} w(t)$ both exist.

Here, "exist" means "exist for u & v".

E.g., a possible u :



Suppose $\overline{W}(t) = \overline{U}(t) + i\overline{V}(t)$ is s.t.

$$\overline{W}' = w \text{ on } [a,b].$$

Then the <u>fundamental theorem of calculus</u> holds, in the form

$$\boxed{\int_a^b w(t)\,dt = \overline{W}(b) - \overline{W}(a)}.$$

The following <u>estimate</u> is crucial.

Suppose $w = u + iv$ is pwc (piecewise cts) on $[a,b]$.

Then: $$\boxed{\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt. \quad (3)}$$

<u>Pf</u>: If $\int_a^b w(t)\,dt = 0$, LHS of (3) $= 0$, & RHS of (3) ≥ 0, so done.

otherwise: $\exists\, r > 0$ & $\theta_0 \in \mathbb{R}$ s.t.

$$\int_a^b w(t)\,dt = re^{i\theta_0} \quad (3)'$$

$$\Rightarrow \left|\int_a^b w(t)\,dt\right| = r \quad (4)$$

$(3)' \cdot e^{-i\theta_0} \Rightarrow r = \int_a^b e^{-i\theta_0} w(t)\,dt$

$$= \mathrm{Re}\left(\int_a^b e^{-i\theta_0} w(t)\,dt\right) \text{ since } r \in \mathbb{R}$$

$$= \int_a^b \mathrm{Re}\left(e^{-i\theta_0} w(t)\right) dt. \quad (5)$$

$$\mathrm{Re}\left(e^{-i\theta_0} w(t)\right) \leq |e^{-i\theta_0} w(t)| = |w(t)|.$$

So, via (4) & (5): $\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$,

showing (3).

Roughly speaking (will be made precise later):
A contour is a parametrised curve in $\mathbb{C}$.
Given $x(t), y(t)$ cts $[a,b] \to \mathbb{R}$,
$z(t) = x(t) + iy(t)$ defines an arc.
This is both a set of points in $\mathbb{C}$ (namely, the image $z([a,b])$, called the trace of the arc), and also a 'recipe for parametrising it.
(cf. B-C p.120 (8Ed p.122), which is a bit sloppy.