LECTURE 16

Def: $f : \Omega \to \mathbb{C}$ is analytic at z_0 if it is differentiable in a nbhd of that pt.
($\Omega \subseteq \mathbb{C}$)

A f^n is singular at z_0 if it is NOT analytic at z_0, but is analytic at some pt in every nbhd of z_0.

E.g. $z \mapsto 1/z$ is analytic on $\mathbb{C}_*$, & is singular at 0.

A f^n is entire if it is analytic on $\mathbb{C}$, e.g. polynomials, e^z, sin, cos, sinh, cosh, etc.

Note: if a f^n is differentiable at precisely one point, it is not analytic there or anywhere, e.g., $z \mapsto |z|^2$

Further on derivatives:

$\frac{d}{dz}(\log z)$: $|z| > 0$

(maybe "$\frac{d}{dz} \log z$" would be a better starting point, as log is multi-valued).

Recall $\operatorname{Log} z = \ln|z| + i \arg z$

$$\underset{re^{i\theta}}{\overset{\parallel}{\operatorname{Log} z}} = \ln r + i\theta$$

$\Rightarrow u = \ln r, \; v = \theta$

* $u_r = 1/r, \; u_\theta = 0, \; v_r = 0, \; v_\theta = 1.$

C/R in polar co-ords: $\begin{cases} * \; r u_r = v_\theta \; ✓ \\ * \; u_\theta = -r v_r \; ✓ \end{cases}$

$\Rightarrow$ sufficient conditions for complex differentiability are satisfied on any subset of $\mathbb{C}_*$ s.t. $\alpha < \theta < \alpha + 2\pi$, for α fixed in $\mathbb{R}$.

On such a subset, Log (or more precisely the single-valued f^n we obtain from Log) is diff^{ble}, &

Lect 15 $\Rightarrow \frac{d}{dz} \log z = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}(1/r + 0)$$

$$= \frac{1}{re^{i\theta}} = 1/z.$$

E.g.: $\frac{d}{dz} \operatorname{Log} z = 1/z$ for $-\pi < \operatorname{Arg} z < \pi$, $|z| \neq 0$.

RMK: cf. p.94 (8Bd p.96), cf. Lec. 9

Log is a branch of Log.

* for $f(z) = z^c$ $c \in \mathbb{C}_*$ fixed, defined on $\mathbb{C}_*$:

$$f(z) = \exp(c \log z) \quad \&$$

$$f'(z) = \exp(c \log z) \cdot c/z \quad (*)$$

$$= z^c \cdot c/z = cz^{c-1} \quad (**)$$

(**) is valid on any domain of the form $\{z : |z| > 0,\ \alpha < \arg z < \alpha + 2\pi\}$, due to the need to choose a branch of log in (*).

<u>RMK</u>: Try for $g(z) = c^z$. ☆

Notation from real analysis

$\Omega \subseteq \mathbb{R}^n \quad n \geq 1$.

* $C(\Omega) = C^0(\Omega) = \{$ cts f^ns $: \Omega \to \mathbb{R}\}$.
* $C^k(\Omega) = \{f^n$s $f : \Omega \to \mathbb{R}$ s.t. f & all its derivatives/partial derivatives of order $\leq k$ exist & are cts on $\Omega\}$.

Note: 0th order derivative of f is f.

* $C^\infty(\Omega) = \{f : \Omega \to \mathbb{R}$ s.t. f & all derivs/partials of all orders exist & are cts$\}$, a.k.a. smooth f^ns.

$f \in C^\omega(\Omega)$ says, at every pt. in $x_0 \in \Omega$:

(i) f has a power series about x_0 (namely, its Taylor series); and

(ii) f is given by its power series, i.e, the power series converges to f on some nbhd of x_0.

$C^\omega(\Omega) =$ real analytic f^ns on Ω.

(i) $\Rightarrow f \in C^\infty$;

(i) $\not\Rightarrow$ (ii) in $\mathbb{R}^n$.

Note: $f(x) = |x|$ is in $C^0(\mathbb{R})$, not in $C^1(\mathbb{R})$.
Using f^ns o.t.f. $x \mapsto x^{3/2}$, $x \mapsto x^{5/2}$ etc,
you can show $C^m(\Omega) \subsetneq C^{m-1}(\Omega)$ $m \geq 2$.

Consider $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$

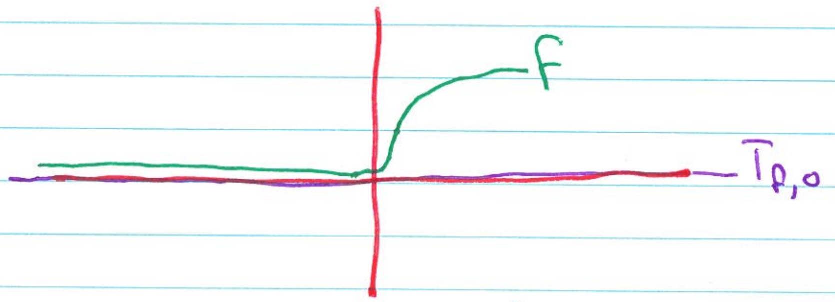
Check: * $f^{(n)}(x)$ exists $\forall x \neq 0$
nth derivative of f

* $f^{(n)}(0) = 0 \ \forall n$

* $f^{(n)}$ is cts on $\mathbb{R}$.

$\Rightarrow$ Taylor series for f at 0 is

$$T_{f,0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0.$$



f is not equal to $T_{f,0}$ on any nbhd of 0.
$f \in C^\infty(\mathbb{R})$, $f \notin C^\omega(\Omega)$.

$$C^\omega \subsetneq C^\infty \subsetneq \cdots C^{1337} \subsetneq C^{1336} \subsetneq \cdots \subsetneq C^2 \subsetneq C^1 \subsetneq C^0$$

§41 – 43 (8 Ed §37–39) Integration.

Consider a $\mathbb{C}$-valued fn. of a real variable

$w(t) = u(t) + i v(t)$. $t \in \mathbb{R}$

Define: $w'(t) = u'(t) + i v'(t)$.

Standard differentiation laws for fns of a $\mathbb{R}$-variable apply:

* $(cw)' = cw'$ $c \in \mathbb{C}$.
* $(w_1 \pm w_2)' = w_1' \pm w_2'$
* $\frac{d}{dt}(e^{ct}) = ce^{ct}$ etc $c \in \mathbb{C}$.
* product, quotient rules etc.

Definite & indefinite integrals of such fns:

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \quad ⊙ \quad a,b \in \mathbb{R}$$

$$\Rightarrow \operatorname{Re}\left(\int_a^b w(t)\,dt\right) = \int_a^b \operatorname{Re}(w(t))\,dt \quad ① \ \&$$

$$\operatorname{Im}\left(\int_a^b w(t)\,dt\right) = \int_a^b \operatorname{Im}(w(t))\,dt \quad ②$$

$\int_a^\infty w(t)\,dt$ etc. are defined analogously.

⊙ certainly makes sense for cts w, i.e. $w \in C^0([a,b])$.