LECTURE 17 Part I

Indeed, ⊙ from Lec. 16 is ok for so-called piecewise cts f^s on $[a,b]$, i.e.,

* $u, v : [a,b] \to \mathbb{R}$:

$\exists\ c_1 < c_2 < \cdots c_n \in (a,b)$ s.t. :

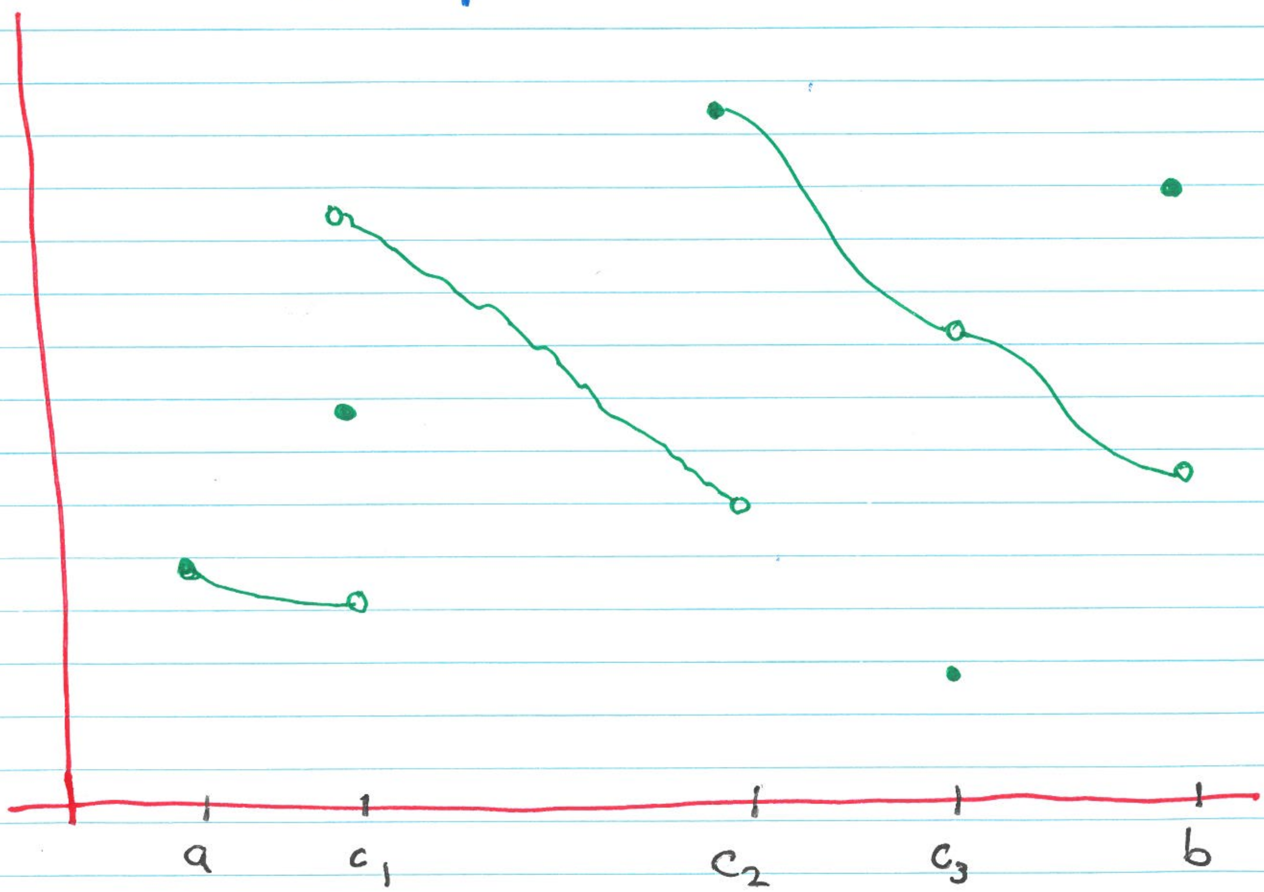
(i) $w = u + iv$ is cts on $(a, c_1), (c_1, c_2), \cdots (c_{n-1}, c_n), (c_n, b)$

(ii) $\lim_{t \to c_j^-} w(t)$, $\lim_{t \to c_j^+} w(t)$ both exist, for $j = 1, \cdots, n$ (they may or may not coincide);

(iii) $\lim_{t \to a^+} w(t)$, $\lim_{t \to b^-} w(t)$ both exist.

Here, "exist" means "exist for u & v".

e.g., a possible u



Suppose $W(t) = U(t) + iV(t)$ s.t.

$W' = w$ on $[a,b]$.

Then the fundamental theorem of calculus holds, in the form

$$\int_a^b w(t)\,dt = W(b) - W(a).$$

The following estimate is crucial.

Suppose $w = u + iv$ is pwc (piecewise cts) on $[a,b]$. Then:

$$\boxed{\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt \quad (3)}$$

PF: If $\int_a^b w(t)\,dt = 0$, LHS of (3) $= 0$, RHS of (3) ≥ 0, so done.

Otherwise: $\exists r > 0$ & $\theta_0 \in \mathbb{R}$ s.t.

$$\int_a^b w(t)\,dt = re^{i\theta_0} \quad (3)'$$

$$\Rightarrow \left|\int_a^b w(t)\,dt\right| = r \quad (4)$$

$(3)' \cdot e^{-i\theta_0} \Rightarrow r = \int_a^b e^{-i\theta_0} w(t)\,dt$

$= \mathrm{Re}\left(\int_a^b e^{-i\theta_0} w(t)\,dt\right)$ since $r \in \mathbb{R}$

$= \int_a^b \mathrm{Re}\left(e^{-i\theta_0} w(t)\right) dt \quad (5)$

$\mathrm{Re}\left(e^{-i\theta_0} w(t)\right) \leq \left|e^{-i\theta_0} w(t)\right| = |w(t)|$.

So via (4) & (5): $\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$, showing (3) $\square$

Roughly speaking (will be made precise later):
A contour is a parametrised curve in $\mathbb{C}$.
Given $x(t), y(t)$ cts : $t \in [a,b] \to \mathbb{R}$.
$z(t) = x(t) + iy(t)$ $a \leq t \leq b$ defines an arc.

This is both a set of points in $\mathbb{C}$ (namely, the image $z([a,b])$, called the trace of the arc, and also a recipe for parametrising it.
(cf. B–C p. 120 (8 Ed p. 122), which is a bit sloppy).