LECTURE 18 part 2.

Contour Integrals : contour C

$\int_C f(z)\,dz$, or $\int_{z_1}^{z_2} f(z)\,dz$, the latter being acceptable if:

a) we know the integral is independent of the path from z_1 to z_2; or

b) the path is understood.

Suppose the contour C is specified by $z(t)$, $z_1 = z(a)$ & $z_2 = z(b)$, $a \le t \le b$, & suppose f is piecewise cts (pwc) on C.

Then $$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.}$$

(cf. Line integrals).

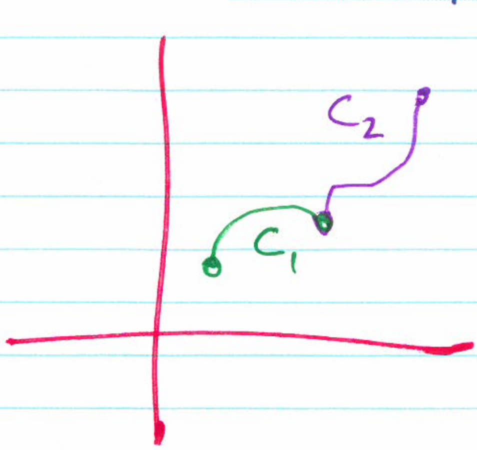
Properties :

linearity * $\int_C (\alpha f)(z)\,dz = \alpha \int_C f(z)\,dz$; $\alpha \in \mathbb{C}$.

* $\int (f+g)(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

* Independent of parametrisation (cf end of Lec 17).

$C_1 + C_2$ defines a contour iff the end pt of C_1 is the start pt of C_2.



Given a contour C, we define a contour $-C$ as follows:

$$w(t) = z(-t) \qquad -b \leq t \leq -a.$$

Then (check with change of parameter formula from Lec 17):

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz.$$

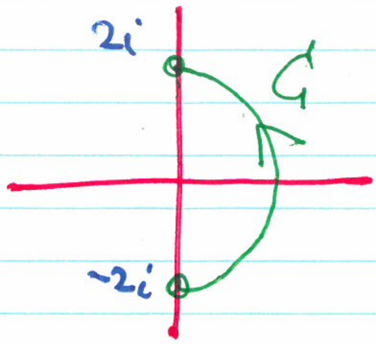
Hence $\underset{\substack{\parallel \\ C_1 + (-C_2)}}{C_1 - C_2}$ is defined iff

start pt of $-C_2$ = end pt of C_1

$\parallel$

end pt of C_2

Ex 1 Evaluate $I = \int_C \bar{z}\, dz$



$C: z = 2e^{i\theta}$

$-\pi/2 \le \theta \le \pi/2$

C is p.w.c. wrt this parametrisation & f is cts on C (cts on $\mathbb{C}$).

$$z'(\theta) = 2ie^{i\theta}$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} f(z(\theta))\, z'(\theta)\, d\theta = \int_{-\pi/2}^{\pi/2} \overline{(2e^{i\theta})} \cdot 2ie^{i\theta} d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} \underbrace{e^{-i\theta} e^{i\theta}}_{1} d\theta = 4\pi i. \quad (*)$$

On C, note $|z| = 2 \Rightarrow z\bar{z} = 4$, $\bar{z} = 4/z$.

So $(*) \Rightarrow$ $\boxed{\int_C \frac{dz}{z} = \pi i}$

See §45 (8th ed §41) for more examples, also problem sheets.