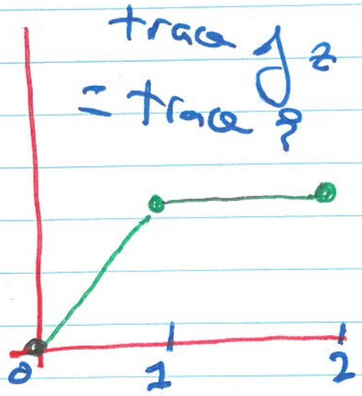
LECTURE 18

Ex 1

$$z(t) = \begin{cases} t + it & 0 \le t \le 1 \\ t + i & 1 < t \le 2 \end{cases}$$

trace γ_z = trace ξ

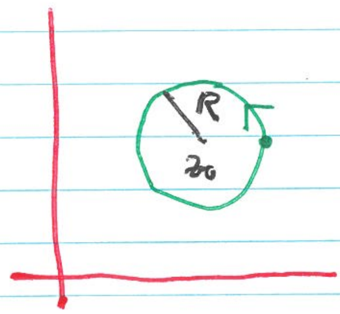


$$\xi(t) = \begin{cases} 2t + 2i & 0 \le t \le \tfrac{1}{2} \\ 1 + i & \tfrac{1}{2} < t \le 1 \\ t + i & 1 < t \le 2 \end{cases}$$

Ex 2

$z = z_0 + Re^{i\theta}$

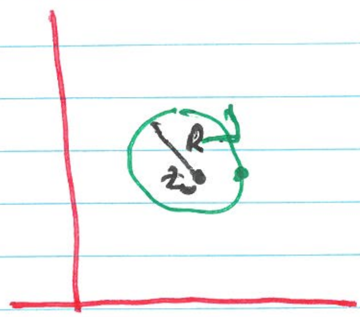
$0 \le \theta \le 2\pi$



Ex 3

$z = z_0 + Re^{-i\theta}$

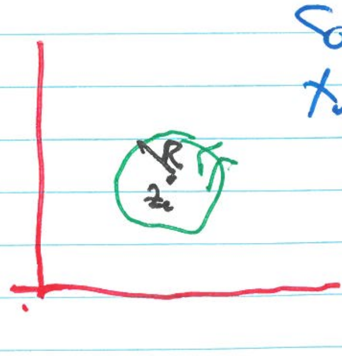
$0 \le \theta \le 2\pi$



Ex 3

$z = z_0 + Re^{2i\theta}$

$0 \le \theta \le 2\pi$



"cover twice"

arcs & curves.

Jordan arc a.k.a. simple arc:
arc s.t. $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$.
(here $z(t) = x(t) + iy(t)$, $t \in [a,b]$).

Jordan curve a.k.a. simple closed curve:

1. $z(b) = z(a)$
2. otherwise $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2 \in [a,b]$.

Ex 1 is a Jordan arc;
Exs 2 & 3 are Jordan curves;
Ex 4 is neither.

Note: Ex 2, 3 & 4 have the same trace.

An arc/curve is called differentiable if
$z'(t)$ exists ($\forall t \in (a,b)$ for a Jordan arc;
$\forall t \in [a,b]$ for a Jordan curve).

If z' is also cts on $[a,b]$, then
$\int_a^b |z'(t)|\, dt$ exists, & defines the arc
length.

Note: If $z(t)$ is a parametrisation of the image arc, we can define another one by

$t = \Phi(\tau)$ with $\Phi(\alpha) = a$ & $\Phi(\beta) = b$,

s.t. $\Phi \in C([\alpha,\beta])$ & $\Phi' \in C([\alpha,\beta])$,

i.e., $\Phi \in C^1([\alpha,\beta])$

So $z(t) = Z_1(\tau) = z(\Phi(\tau))$.

Assume $\Phi'(\tau) > 0 \quad \forall \tau$

$$\int_a^b |z'(t)|\, dt = \int_\alpha^\beta |z'(\Phi(\tau))| \Phi'(\tau)\, d\tau$$

$$= \int_\alpha^\beta |Z_1'(\tau)|\, d\tau.$$

i.e. arc length is independent of parametrisation.

A contour is an arc/(e.g., simple closed curve) s.t.:

(I) z is cts;

(II) piecewise differentiable, i.e., z' is piecewise cts.

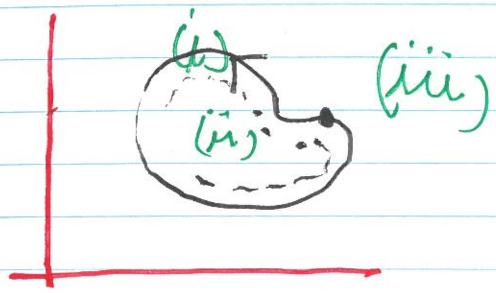
If initial & final values of z coincide & there are no other self intersections, we have a simple closed contour.

Jordan curve Thm: Any simple closed contour divides $\mathbb{C}$ into 3 disjoint sets:

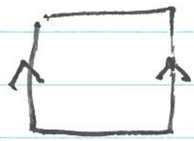
(i) on the curve;

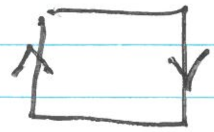
(ii) inside the curve;

(iii) outside the curve.



cylinder.





Möbius strip.

Rmk:

Statement still holds if we remove assumption (II): CARE!