LECTURE 20

Anti-differentiation BC §48-49 (8th ed §44-45)

THM: Let D be a domain in $\mathbb{C}$ (i.e., D is an open, connected subset of $\mathbb{C}$). Let f be cts on D. An anti-derivative of f on D is F s.t.

$$F'(z) = f(z) \qquad \forall z \in D.$$

The following are equivalent:

(i) f has an anti-derivative on D;

(ii) for any $z_1, z_2 \in D$ & any contour from z_1 to z_2 in D we have that $\int_C f(z)\,dz$ is independent of C;

(iii) for any closed contour C in D, there holds:

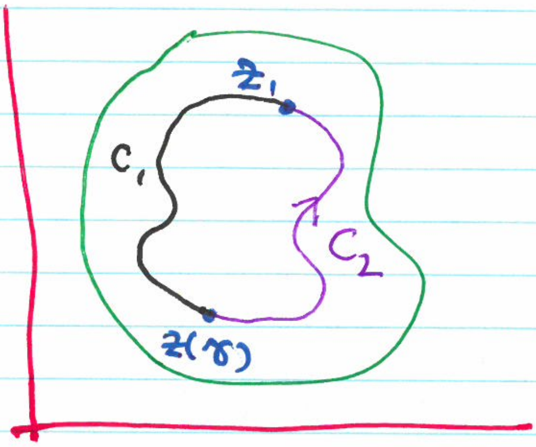
$$\int_C f(z)\,dz = 0.$$

PF: (i) $\Rightarrow$ follows from the fundamental thm (see BC).

(ii) $\Rightarrow$ (iii) Let C be a closed contour in D with $z(a) = z(b) = z_1$.
Fix $\gamma \in (a,b)$ s.t. $z(\gamma) \neq z_1$, & define contours in D:

C_1 $\quad w(t) = z(t) \quad a \le t \le \gamma$;

C_2 $\quad \tilde{w}(t) = z(t) \quad \gamma \le t \le b$.



$C = C_1 + C_2$

Since $C = C_1 + C_2$, there holds:

$$\int_C f = \int_{C_1 + C_2} f$$

$$= \int_{C_1} f + \int_{C_2} f$$

$$= \int_{C_1} f - \int_{-C_2} f \quad \text{by def}^{n} \text{ of } -C_2.$$

$= 0$ by (ii) (since C_1 & $-C_2$ are both contours from z_1 to $z(\gamma)$ in D).

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) : see BC.

Note in particular for $C : z_1 \to z_2$ in D under (i) – (iii), there holds:

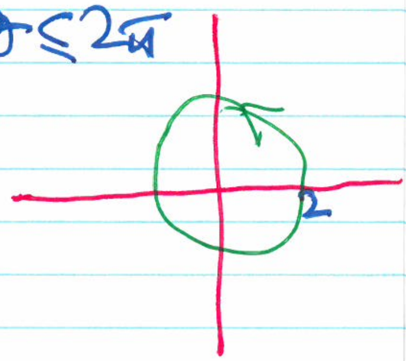
$$\int_C f(z)\,dz = F(z(b)) - F(z(a))$$ where F is <u>any</u> antiderivative of f, for a contour from $z(a)$ to $z(b)$ in D.

<u>Further examples of contour integrals.</u>

<u>Ex 2</u> $\quad I_2 = \int_0^{1+i} z^2\,dz$.

$f(z) = z^2$ is cts in $\mathbb{C}$, so $\int_C z^2\,dz$ is defined for any path C from 0 to $1+i$, & $F(z) = z^3/3$ is an anti-derivative on $\mathbb{C}$, so I_2 is well-defined (independent of path) by Th^m, & from above $I_2 = F(1+i) - F(0) = \frac{2}{3}(-1+i)$.

<u>Ex 3</u> $\quad I_3 = \int_C \frac{dz}{z^2}$, $\quad C = 2e^{i\theta}$, $0 \le \theta \le 2\pi$



$f(z) = 1/z^2$ has an anti-derivative on $\mathbb{C}_*$, namely $-1/z$, & C is a contour lying entirely in $\mathbb{C}_*$, so Th^m $\Rightarrow I_3 = 0$.

Same argument shows

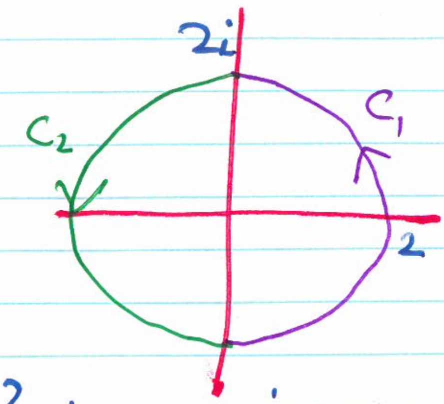
$$\boxed{\int_C z^n\,dz = 0 \quad \forall n \in \mathbb{Z} \setminus \{-1\}.}$$

Ex 4 $I_4 = \int_C \frac{dz}{z}$ $\quad C = 2e^{i\theta} \quad 0 \le \theta \le 2\pi$

Cannot repeat the argument of Ex. 3 directly.

Instead: $I_4 = I_{41} + I_{42}$

where $I_{4j} = \int_{C_j} f \quad j = 1, 2.$



On $D = \mathbb{C} \setminus \{-\text{ve Real axis} \cup \{0\}\}$, Log z is a primitive (anti-derivative) of $1/z$, & $C_1 \subset D$.

So Th^m $\Rightarrow I_{41} = \text{Log}(2i) - \text{Log}(-2i)$

$$= \ln|2i| + i\tfrac{\pi}{2} - \left(\ln|-2i| + i\left(-\tfrac{\pi}{2}\right)\right)$$

$$= \pi i.$$ (Rmk: agrees with calculation from Lec 18, Ex 1).

For I_{42}: on $D' = \mathbb{C} \setminus \{+\text{ve Real axis} \cup \{0\}\}$ $1/z$ has a primitive, e.g., $\mathcal{L}og$, given by

$$\mathcal{L}og(z) = \ln|z| + i\,\mathcal{A}rg\, z, \quad 0 < \mathcal{A}rg < 2\pi$$

Note $C_2 \subset D$.

So Th^m $\Rightarrow I_{42} = \mathcal{L}og(-2i) - \mathcal{L}og(2i)$

$$= \ln 2 + i\tfrac{3\pi}{2} - \left(\ln 2 + i\tfrac{\pi}{2}\right)$$

$$= i\pi.$$

$$I_4 = I_{41} + I_{42} = 2\pi i.$$

Note same result for a circle of any (+ve) radius centre zero, also for previous example.

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So $\int_C z^n\,dz = \begin{cases} 0 & n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i & n = -1 \end{cases}$

for any circle C centred at 0, positively oriented.

§50 (8Ed) Cauchy-Goursat

Let C be a simple closed contour in $\mathbb{C}$. If f is analytic on C & its interior, then

$$\boxed{\int_C f(z)\,dz = 0}$$

Rmk $\int_C f(z)\,dz = 0 \not\Rightarrow f$ is analytic in & on C : consider e.g. $\int_C z^n\,dz = 0$, $n = -2, -3, -4, \ldots$ for any circle C centred at 0.

Pf: (1) Do it for  ;

(2) Approximate.