LECTURE 20

Rmk re diff$^{\text{ble}}$. curves.

Consider $\gamma(t) = \cos t + \sin(|t-\pi|)i \qquad 0 \leq t \leq 2\pi$

$$= \begin{cases} \cos t + i\sin(\pi - t) & 0 \leq t \leq \pi \\ \cos t + i\sin(t-\pi) & \pi < t \leq 2\pi \end{cases}$$

γ is cts on $[0,2\pi]$, $\gamma(0) = \gamma(2\pi)$, γ cts at π.

$$\gamma'(t) = \begin{cases} -\sin t - i\cos(\pi - t) & 0 < t < \pi \\ -\sin t + i\cos(t-\pi) & \pi < t < 2\pi \end{cases}$$

$\gamma'_-(\pi) = -i$.

$\gamma'_+(\pi) = i$.

$\gamma'_-(2\pi) = -i$.

$\gamma'_+(0) = i$.

close to an example of γ that is differentiable on $[a,b]$, but not a C' curve. T.B.C. ...

Contour Integrals:

$\int_C f(z)\,dz$ or $\int_{z_1}^{z_2} f(z)\,dz$, the latter being acceptable if:

a) we know the integral is independent of the path from z_1 to z_2; or

b) if the path is understood.

Suppose the contour is specified by $z(t)$, $z_1 = z(a)$ & $z_2 = z(b)$, $a \le t \le b$, & suppose f is piecewise cts (pwc) on C.

Then:

$$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt}.$$

cf. line integrals.

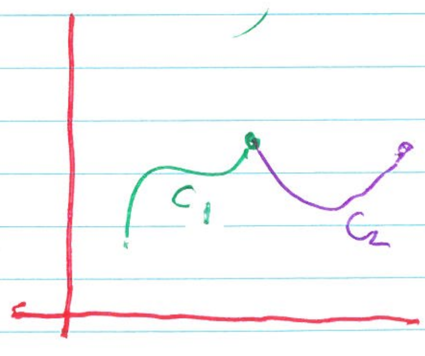
Properties:

* linearity $\int_C (\alpha f)(z)\,dz = \alpha \int_C f(z)\,dz$; $\alpha \in \mathbb{C}$.

$$\int_C (f+g)(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.$$

* independent of the parametrisation (cf. end of Lec. 18).

$C_1 + C_2$ defines a contour when the end pt of C_1 is the start point of C_2.



Given a contour C_1, define a contour $-C_1$ as follows:

$$w(t) = z(-t), \quad -b \leq t \leq -a.$$

Then (check with change of parameter formula from Lec 18):

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

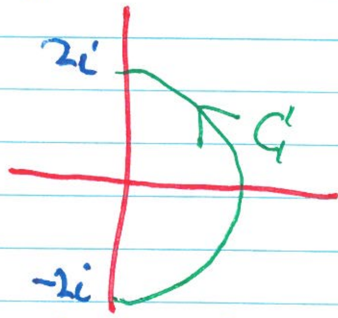
Hence, $C_1 - C_2$ is defined when

$$C_1 - C_2 = C_1 + (-C_2)$$

the start point of $-C_2$ = end pt of C_1

i.e., the end pt of C_2 = end pt of C_1.

Ex 1 Evaluate $I = \int_C \bar{z}\,dz$



$C: z = 2e^{i\theta}$

$-\pi/2 \leq \theta \leq \pi/2$

C is p.w.c. w.r.t. this parametrisation, & f is cts on C (cts on $\mathbb{C}$).

Note $z'(\theta) = 2ie^{i\theta}$

$$I = \int_{-\pi/2}^{\pi/2} f(z(\theta))\, z'(\theta)\,d\theta = \int_{-\pi/2}^{\pi/2} \overline{(2e^{i\theta})} \cdot 2ie^{i\theta}\,d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} \underbrace{e^{-i\theta} e^{i\theta}}_{1}\,d\theta = 4\pi i \quad *$$

On C, note $|z| = 2 \Rightarrow z\bar{z} = 4 \Rightarrow \bar{z} = 4/z$.

So (*) $\Rightarrow$ $\boxed{\int_C \frac{dz}{z} = \pi i}$.

See §45 (8 Ed §41) for more examples, also Worksheets.

Anti-differentiation BC §48-49 (8 Ed §44-45)

THM: Let D be a domain in $\mathbb{C}$ (i.e., D is an open, connected subset of $\mathbb{C}$). Let f be cts on D.

An antiderivative of f on D is F such that $F'(z) = f(z)$ on D.

The following are equivalent:

(i) f has an anti-derivative on D;

(ii) For any z_1 & $z_2 \in D$ & any contour from z_1 to z_2 in D we have that $\int_C f(z)dz$ is independent of C;

(iii) For any closed contour C in D, there holds: $\int_C f(z)dz = 0$.

PF: (i) $\Rightarrow$ (ii) follows from Fundamental th^m (see BC).

(ii) $\Rightarrow$ (iii) Lec 21.