LECTURE 21

Anti-differentiation   BC §48-49 (8Ed 44-45)

THM Let D be a domain in $\mathbb{C}$ (i.e., D is an open, connected subset of $\mathbb{C}$). Let f be cts on D.

An anti-derivative of f on D is F s.t.

$$F'(z) = f(z) \quad \text{on } D.$$

The following are equivalent:

(i) f has an anti-derivative on D;

(ii) for any z_1, z_2 & any contour C from z_1 to z_2 in D we have that $\int_C f(z)\,dz$ is independent of C;

(iii) For any closed contour C in D, there holds: $\int_C f(z)\,dz = 0$.

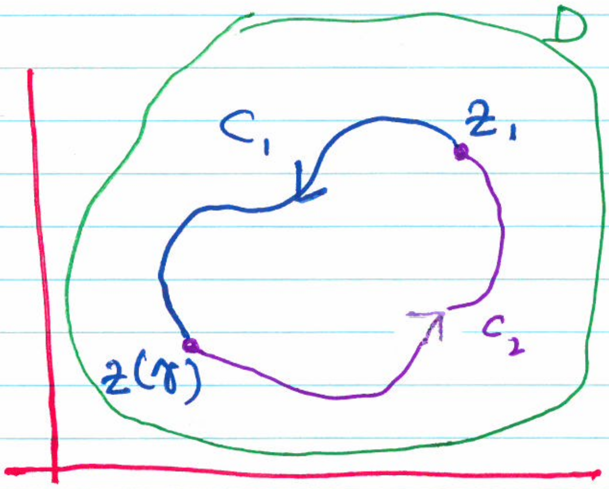
Pf: (i) $\Rightarrow$ (ii) follows from fundamental th^m (see BC).

(ii) $\Rightarrow$ (iii) Let C be a closed contour in D with $z(a) = z(b) = z_1$.

Fix $\gamma \in (a,b)$ s.t. $z(\gamma) \neq z_1$, & define contours in D:

$C_1 \quad w(t) = z(t) \quad a \leq t \leq \gamma$;

$C_2 \quad \tilde{w}(t) = z(t) \quad \gamma \leq t \leq b$.



$C = C_1 + C_2$.

Since $C = C_1 + C_2$, there holds

$$\int_C f = \int_{C_1+C_2} f$$

$$= \int_{C_1} f + \int_{C_2} f$$

$$= \int_{C_1} f - \int_{-C_2} f \qquad \text{by def}^{n} \text{ of } -C_2$$

$$= 0 \text{ by (ii)}$$

(C_1 & C_2 are both paths from z_1 to $z(\gamma)$ in D).

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) : see BC.

Note in particular for $C: z_1 \to z_2$ in D under (i)–(iii), there holds:

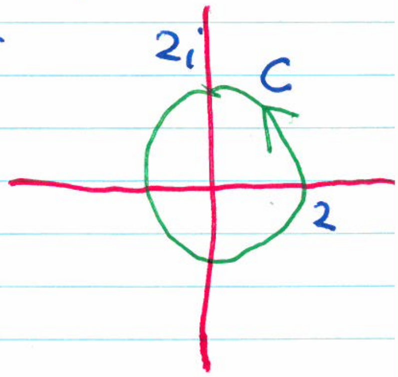
$\int_C f(z) = F(z(b)) - F(z(a))$ where F is any antiderivative of f, for a contour from $z(a)$ to $z(b)$ in D.

Further examples of contour integrals

Ex 2 $\quad I_2 = \int_0^{1+i} z^2\, dz$

$f(z) = z^2$ is cts in $\mathbb{C}$, so $\int_C z^2\, dz$ is defined for any path C from 0 to $1+i$, & $F(z) = z^3/3$ is an anti-derivative on $\mathbb{C}$, So I_2 is well defined (independent of path) by Thm, & from above, $I_2 = F(1+i) - F(0)$
$= \frac{2}{3}(-1+i)$.

Ex 3 $\quad I_3 = \int_C \frac{dz}{z^2} \qquad C = 2e^{i\theta},\ 0 \le \theta \le 2\pi$



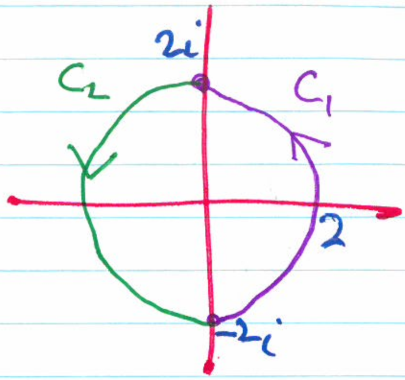
$f(z) = 1/z^2$ has an anti-derivative on $\mathbb{C}_*$, namely $-1/z$, & C is a contour lying entirely in $\mathbb{C}_*$, so Thm $\Rightarrow I_3 = 0$.

Same argument shows:

$$\boxed{\int_C z^n dz = 0 \qquad \forall n \in \mathbb{Z} \setminus \{-1\}.}$$

Ex 4: $I_4 = \int_C \frac{dz}{z}$ $\quad C = 2e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$

Can't repeat the argument of Ex 3 directly.



Instead: $I_4 = I_{41} + I_{42}$,

where $I_{4j} = \int_{C_j} f \quad j = 1,2$.

On $D = \mathbb{C} \setminus \{\text{-ve Real axis} \cup \{0\}\}$, Log z is a primitive (anti-derivative) of $1/z$, & $C_1 \subset D$,

So Thm $\Rightarrow I_{41} = \mathrm{Log}(2i) - \mathrm{Log}(-2i)$

$$= \ln|2i| + i\tfrac{\pi}{2} - \left(\ln|-2i| + i\left(-\tfrac{\pi}{2}\right)\right)$$

$$= \pi i.$$ (rmk: agrees with calculation from Lec 19, Ex 1.

For I_{42}: on $D' = \{\mathbb{C} \setminus \{\text{+ve Real axis} \cup \{0\}\}\}$, $1/z$ has a primitive, e.g. $\mathcal{L}og$, given by

$$\mathcal{L}og\, z = \ln|z| + i\,\mathrm{Arg}\, z \quad , \quad 0 < \mathrm{Arg} < 2\pi$$

Note $C_2 \subset D'$.

So Thm $\Rightarrow I_{42} = \mathcal{L}og(-2i) - \mathcal{L}og(2i)$

$$= \ln 2 + i\tfrac{3\pi}{2} - \left(\ln 2 + i\tfrac{\pi}{2}\right)$$

$$= i\pi$$

$\Rightarrow I_4 = I_{41} + I_{42} = 2\pi i$

Note: same result for a circle of any (+ve) radius centre zero, also for previous example.

So $$\boxed{\int_C z^n \, dz = \begin{cases} 0 & n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i & n = -1 \end{cases}}$$

for any circle C centred at the origin, positively oriented.

Next: Cauchy-Goursat.