

LECTURE 21

Key step in the proof of Cauchy-Goursat is the M-L estimate:

Suppose f is cts on a contour C given by $z(t)$, $a \leq t \leq b$.

Then $\exists M$ s.t. $|f(z)| \leq M \quad \forall z \in C$.

This follows since $f \circ z : [a, b] \rightarrow \mathbb{C}$ is cts.

Hence, $|f \circ z| : [a, b] \rightarrow \mathbb{R}$ is cts, so we can apply the extreme value theorem (in $\mathbb{R}$) from MATH 2400/2401.

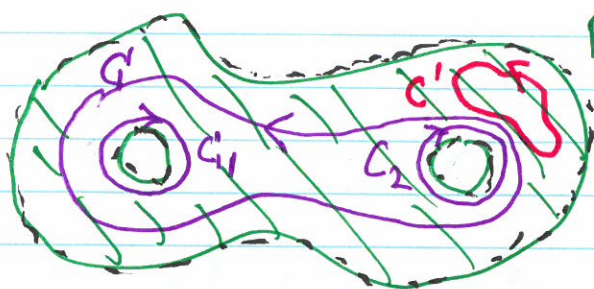
$$\begin{aligned} \text{So, } \left| \int_C f(z) dz \right| &= \left| \int_a^b f(z(t)) z'(t) dt \right| \\ &\leq \int_a^b |f(z(t))| |z'(t)| dt \\ &\leq M \int_a^b |z'(t)| dt \\ &= M \cdot l(C). \end{aligned}$$

This is the M-L estimate: $\int_C f(z) dz \leq M l_C$

RMK: A domain D is simply connected if, for every simple closed contour G in D , there holds $\text{Int } G \subset D$, i.e., "no holes" (i.e., all simple closed contours in D are null homotopic in D).



D_1 is simply connected.



D_2 is not s.c. :
e.g. G' is null-homotopic,
but e.g. G, G_1, G_2 are
not.

Extension of Cauchy-Goursat to multiply-connected domains.

Let f be analytic on simple closed contours G & $G_1, \dots, G_n \subset \text{Int } G$, $G_1, \dots, G_n$ disjoint, & on the interior of the domain bdd by $G, G_1, \dots, G_n$.

Let G be +vely oriented & $G_1, \dots, G_n$ be negatively oriented, orientations taken in $\mathbb{C}$.

Then:

$$\oint_G f + \sum_{j=1}^n \int_{G_j} f = 0.$$

Cauchy Integral Formula §54 (8Ed §50).

Let f be analytic on & inside a simple closed curve C' that is positively oriented.

(i.e., if $z(t)$ parametrises C' , then as $t \uparrow$, $\text{Int } C'$ stays on the curves LHS).

Then if $z_0 \in \text{Int } C'$, we have

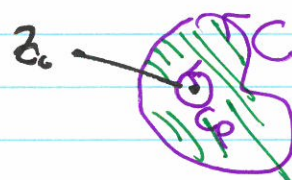
$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz \quad \textcircled{1} \quad \text{i.e.,}$$

$$2\pi i f(z_0) = \int_C \frac{f(z)}{z-z_0} dz. \quad \textcircled{1}'$$

P.F.: set $C_p = \{z(\theta) = z_0 + pe^{i\theta}, 0 \leq \theta \leq 2\pi\}$

for $p > 0$ suff small that

$$\text{Int}(C_p) \subset \Delta = \text{Int } C.$$



Then $z \mapsto \frac{f(z)}{z-z_0}$ is analytic on $\text{Int } C - \text{Int } C_p$ & on C & C_p .

So C - C_p extension $\Rightarrow$

$$\int_C \frac{f(z)}{z-z_0} dz = \int_{C_p} \frac{f(z)}{z-z_0} dz$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0} - f(z_0) \cdot \int_C \frac{dz}{z-z_0} = \int_C \frac{f(z) - f(z_0)}{z-z_0} dz \quad (2)$$

$(*) = 2\pi i$ (cf. Lec 20) $\forall \rho$ suff small.

Since f is analytic at z_0 , it is cts at z_0 .

$\Rightarrow$ Given $\varepsilon > 0 \exists \delta > 0$ s.t.

$$|f(z) - f(z_0)| < \varepsilon \quad \forall |z - z_0| < \delta. \quad (3)$$

Choose $\rho < \delta$.

$$|f(z_0 + \rho e^{i\theta}) - f(z_0)| < \varepsilon \quad \text{😊}$$

So $(2) \Rightarrow$

$$\left| \int_C \frac{f(z)}{z-z_0} - 2\pi i f(z_0) \right| \leq \int_C \frac{|f(z) - f(z_0)|}{|z-z_0|} dz$$

$$= \frac{1}{\rho} \int_C |f(z) - f(z_0)| dz$$

$$< \frac{1}{\rho} \varepsilon \cdot 2\pi \rho \quad \text{via 😊, M-L}$$

$$= 2\pi \varepsilon.$$

Send $\varepsilon \searrow 0 \Rightarrow (1)'$.