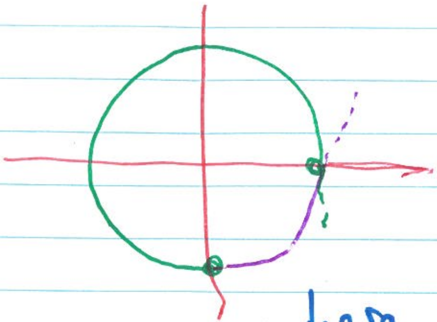
LECTURE 21



put $\gamma(t) = \begin{cases} \cos t + i \sin t & 0 \le t \le \frac{3\pi}{2} \\ x(t) + i y(t) & \frac{3\pi}{2} < t \le 2\pi \end{cases}$

where $x(t) = \left(t - \frac{3\pi}{2}\right) + \frac{4(1-\frac{\pi}{2})}{\pi^2}\left(t - \frac{3\pi}{2}\right)^2$

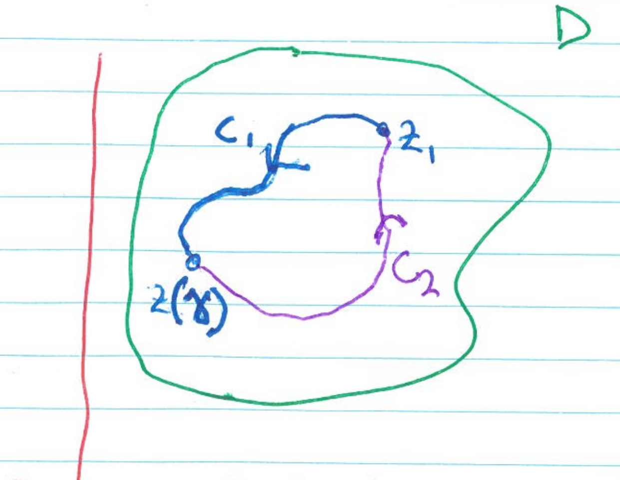
& $y(t) = (x(t))^2 - 1$.

γ satisfies: γ is continuous on $[0,2\pi]$ differentiable on $[0,2\pi]$ but $\gamma_+'(0) \neq \gamma_-'(2\pi)$.

WTS (ii) $\Rightarrow$ (iii) from THM, Lec 20.
Let C be a closed contour in D with $z(a) = z(b) = z_1$.
Fix $\gamma \in (a,b)$ s.t. $z(\gamma) \neq z_1$, & define contours in D:

C_1 $\quad w(t) = z(t) \quad a \le t \le \gamma$;

C_2 $\quad \tilde{w}(t) = z(t) \quad \gamma \le t \le b$.



$C = C_1 + C_2$

Since $C = C_1 + C_2$, there holds:

$$\int_C f = \int_{C_1 + C_2} f$$
$$= \int_{C_1} f + \int_{C_2} f$$
$$= \int_{C_1} f - \int_{-C_2} f \qquad \text{by def}^n \text{ of } -C_2.$$
$$= 0$$

by (ii), since C_1 & $-C_2$ are both paths from z_1 to $z(\gamma)$ in D. □

(iii) ⇒ (ii) ⇒ (i) : see BC.

Note in particular for $C : z_1 \to z_2$ in D under (i)–(iii), there holds:

$$\int_C f(z)\,dz = F(z_2) - F(z_1),$$

where F is <u>any</u> anti-derivative of f.

<u>Further examples of contour integrals.</u>

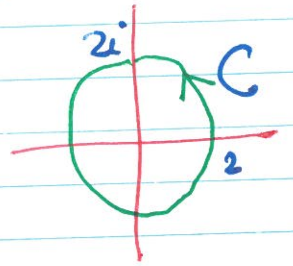
<u>Ex2</u> $\quad I_2 = \int_0^{1+i} z^2\,dz$

$f(z) = z^2$ is cts on $\mathbb{C}$, so $\int_C z^2\,dz$ is defined for any contour C from 0 to $1+i$, & $F(z) = z^3/3$ is an antiderivative on $\mathbb{C}$.

So I_2 is well-defined (independent of path) b Th^m, & by the above,

$$I_2 = F(1+i) - F(0) = \tfrac{2}{3}(-1+i).$$

Ex 3 $I_3 = \int_C \frac{dz}{z^2}$ $\quad C = 2e^{i\theta},\ 0 \le \theta \le 2\pi$

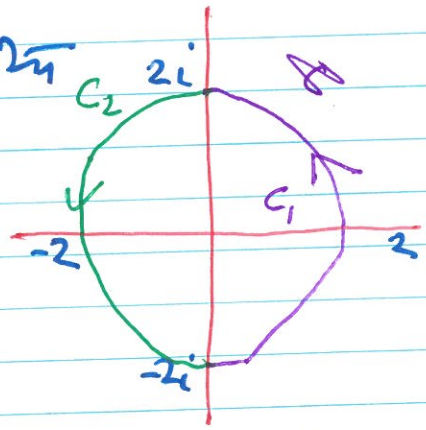


$f(z) = 1/z^2$ has an anti-derivative on $\mathbb{C}_*$, namely $-1/z$ & C is a contour lying entirely in $\mathbb{C}_*$, so Th^m $\Rightarrow$ $I_3 = 0$.

Same argument shows

$$\boxed{\int_C z^n \, dz = 0 \quad \forall n \in \mathbb{Z} \setminus \{-1\}.}$$

Ex 4 $I_4 = \int_C \frac{dz}{z}$ $\quad C = 2e^{i\theta}\ \ 0 \le \theta \le 2\pi$



Can't repeat the argument of Ex 3 directly. Instead:

$I_4 = I_{41} + I_{42}$, where $I_{4j} = \int_{C_j} f$

On $D_1 = \mathbb{C} - \{\text{-ve Real axis} \cup \{0\}\}$, $\operatorname{Log} z$ is a primitive (i.e., an antiderivative) of $1/z$, & $C_1 \subset D_1$, so THM $\Rightarrow$

$$I_{41} = \operatorname{Log}(2i) - \operatorname{Log}(-2i)$$
$$= \ln|2i| + i\tfrac{\pi}{2} - \left(\ln|-2i| + i\left(-\tfrac{\pi}{2}\right)\right)$$
$$= i\pi.$$

(rmk: agrees with direct calculation from Lec 20).

For I_{42}: On $D_2 = \{\mathbb{C} \setminus \{+\text{ve Real axis} \cup \{0\}\}$,
$1/z$ has a primitive, e.g. $\mathcal{L}og$, given by

$\mathcal{L}og\, z = \ln|z| + i\,\mathcal{A}rg\, z \qquad 0 < \mathcal{A}rg\, z < 2\pi$

unique value of $\arg z$

Note $C_2 \subset D_2$

So THM $\Rightarrow$ $I_{42} = \mathcal{L}og(-2i) - \mathcal{L}og(2i)$

$= \ln 2 + i\frac{3\pi}{2} - (\ln 2 + i\frac{\pi}{2})$

$= i\pi$

$\Rightarrow I_4 = I_{41} + I_{42} = 2\pi i$.

Note: same result holds for a circle of any +ve radius, centre 0, ... also for $Ex\,3$.

So $\boxed{\int_C z^n = \begin{cases} 0 & n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i & n = -1 \end{cases}}$

for any circle C centred at the origin, positively (ie, counterclockwise) oriented.