LECTURE 22

Let f be analytic in & on C, a simple closed contour in $\mathbb{C}$, traversed in the +ve sense, & let $z_0 \in \operatorname{Int} C$.

Cauchy (Lec 21) $\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$

There further holds:

Thm §55 (8Ed §52)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n \geq 1 \quad (1)$$

nth derivative of f at z_0.

works for $n=0$
$0! = 1$, $f^{(0)} = f$.

Pf: see ex 9 §57 (8Ed Ex 9 §52)

Thm §57 If f is analytic at z_0, then its derivatives of all orders exist & are analytic at z_0.

Pf: f is analytic at $z_0 \Rightarrow f$ is analytic on $B_\varepsilon(z_0)$ for some $\varepsilon > 0$. Then for

$C = \{z_0 + \frac{\varepsilon}{2} e^{i\theta}, 0 \leq \theta \leq 2\pi\}$, f is analytic on C & in $\operatorname{Int} C$, so

$$(1) \Rightarrow f''(z) = \frac{1}{\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^3} d\zeta \quad \forall z \in \operatorname{Int} C.$$

$\Rightarrow f''$ exists on $\operatorname{Int} C \Rightarrow f'$ is analytic on $\operatorname{Int} C$, & at z_0 in particular.

Reapply to f' to show f'' is analytic, etc. $\square$

cf. the situation in $\mathbb{R}$, e.g. $f(x) = |x|^3$:
f, f', f'' are all cts on $\mathbb{R}$, but $f'''(0)$ does not exist.

RMK: For $f = u + iv$: f is analytic at $z_0 = x_0 + iy_0$ $\Rightarrow$ partials of all orders of u & v exist & are cts at (x_0, y_0).

THM (Morera) Let f be cts on a domain $\Omega \subseteq \mathbb{C}$. If $\int_C f = 0$ $\forall$ closed contours lying in Ω, then f is analytic in Ω.

Pf: By Thm. §48 (Lec. 20), f has a primitive on Ω, say F. But then $F' = f$ exists & is cts on Ω by cond^{n}s of the thm., so F is analytic. So by Thm. §57, $f = (F')$ is also analytic on Ω.

A number of nice results follow from this, & in particular from Th^m $S_3$55.

I. Let f be analytic in & on $C_R(z_0)$ $= \{z_0 + Re^{i\theta},\ 0 \leq \theta \leq 2\pi\}$

Then: $$\boxed{|f^{(n)}(z_0)| \leq \frac{n!\,M_R}{R^n},} \quad (2)$$

where $M_R = \max_{z \in C_R} |f(z)|$.

Pf: note that M_R is well defined (& in particular, finite) by Extreme Value Theorem.

$$|f^{(n)}(z_0)| \overset{\text{Th}^m\, S55}{=} \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right|$$

$$= \frac{n!}{2\pi} \int_{C_R} \frac{|f(z)|}{|z-z_0|^{n+1}}\,dz$$

$|f(z)| \leq M_R$

$|z-z_0|^{n+1} = R^{n+1}$ on C_R.

$$\leq \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \int_{C_R} dz$$

$$\frac{n!}{\cancel{2\pi}} \frac{M_R}{\cancel{R}R^n} \cancel{2\pi R}$$

$$= \frac{n!\,M_R}{R^n}$$

$\square$

Rmk Leads to e.g. Bieberbach conjecture.

II Liouville's Th^m:

If $f: \mathbb{C} \to \mathbb{C}$ is bded & entire, then f is constant.

Pf: Suppose that $|f| < M$ on $\mathbb{C}$, $z_0 \in \mathbb{C}$.

Apply 2 for $n=1$ on $C_R = \{z_0 + Re^{i\theta}, 0 \le \theta \le 2\pi\}$.

$$|f'(z_0)| \le \frac{1!\, M}{R^1} = M/R \quad (**)$$

Letting $R \to \infty$ & noting the LHS of (**) is independent of R, we see there must hold $f'(z_0) = 0$. Since z_0 was arbitrary, f must be constant.

IMPORTANT entire is crucial here.

$\exists$ nonconstant, bded analytic f^{n}s on "large", indeed unbded domains in $\mathbb{C}$.

E.g. on the UHP (upper half plane) $\{z : \mathrm{Im}\, z > 0\}$, consider $z \mapsto e^{iz}$.

For $z = x+iy \in$ UHP, $y > 0$, so

$$|e^{iz}| = |e^{i(x+iy)}| = e^{ix} e^{-y} = e^{-y} < 1.$$

III. Fund th^m of algebra. (Pracs Wk10).

Next: §11.2 conformal maps.