LECTURE 22

§50 (8 Ed §46) Cauchy–Goursat.

Let C be a simple closed contour in $\mathbb{C}$. If f is analytic on C & its interior, then

$$\boxed{\int_C f(z)\,dz = 0.}$$

Pf: Consider first the additional assumptions:

(A) C is +vely oriented;

(B) Assume f' is cts.

Write $f = u + iv$, let C be parametrised by
$z(t) = x(t) + i y(t)$, $a \le t \le b$, $z(a) = z(b)$.

$$\text{So } \int_C f(z)\,dz = \int_C f(z(t))\, z'(t)\,dt$$
$$= \int_a^b (u+iv)(x'+iy')\,dt$$
$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt$$
$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy).$$

Green's

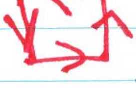
$$= \int_R (-v_x - u_y)\,dx\,dy + i\int_R (u_x - v_y)\,dx\,dy.$$

where $R = \operatorname{Int} C$.

$$= 0 \quad \text{via C/R.}$$

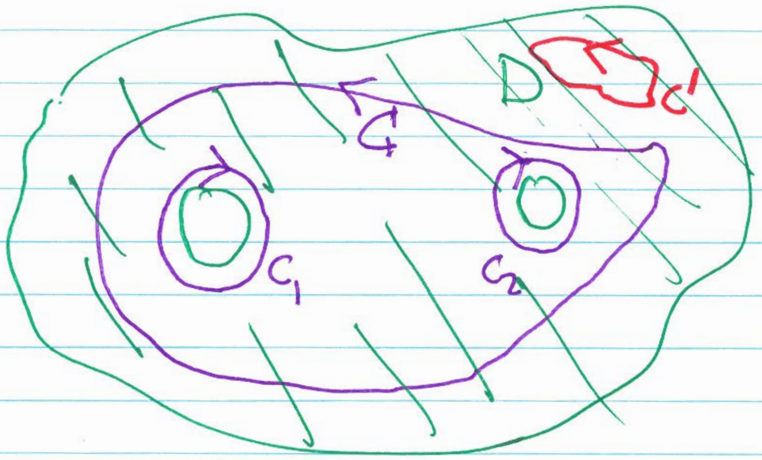
Remove (A) ✓. (Cauchy 1825).

Remove (B) messier (Goursat, 1900).

Idea: Do it for , then approximate.

Note: $\int_C f(z) = 0 \not\Rightarrow f$ is analytic in & on C, e.g., $f(z) = 1/z^2$.

Rmk a domain D is simply connected if, for every simple closed contour C in D there holds $\text{Int}\, C \subset D$, i.e., "no holes", i.e. all simple closed contours in D are null homotopic.



C' is null homotopic in D, & C is not.

Let f be analytic on simple closed contours C & $C_1, \cdots C_n \subset \text{Int}\, C$, $C_1, \cdots, C_n$ disjoint, with disjoint interiors, & f is analytic on the interior of the domain bounded by $C, C_1, \cdots, C_n$.

Let C be +vely oriented & $C_1, \cdots, C_n$ be negatively oriented, orientations taken in $\mathbb{C}$.

Then: $\int_C f + \sum_{j=1}^{n} \int_{C_j} f = 0$.

This is the extension of Cauchy-Goursat to multiply-connected domains.

§54 (8Ed §50)

Cauchy Integral Formula

Let f be analytic on & inside a simple closed curve C that is +vely oriented (i.e., if $z(t)$ parametrises C, then as $t\uparrow$, Int C stays on the LHS of the curve).

Then if $z_0 \in$ Int C, we have

$$\boxed{\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad (1), \text{ i.e.} \\ 2\pi i\, f(z_0) &= \int_C \frac{f(z)}{z-z_0}\,dz \quad (1)' \end{aligned}}$$

Pf: set $C_\rho = \{z(\theta) = z_0 + \rho e^{i\theta}, 0 \le \theta \le 2\pi\}$.

Choose $\rho > 0$ suff small that

Int $(C_\rho) \subset$ Int C $= D$.

Then $z \mapsto \frac{f(z)}{z-z_0}$ is analytic on Int $C \setminus$ Int C_ρ, & on C_ρ & C.

So C-G extension $\Rightarrow$

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz.$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz - f(z_0)\underbrace{\boxed{\int_{C_\rho}\frac{dz}{z-z_0}}}_{*} = \int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}\,dz. \quad (2)$$

$*$ $= 2\pi i$ (cf. Lec 21) $\forall \rho > 0$.

Since f is analytic at z_0, it is cts at z_0. $\Rightarrow$ Given $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$$|f(z) - f(z_0)| < \varepsilon \quad \forall |z - z_0| < \delta \qquad ③$$

Choose $\rho < \delta$

$$\Rightarrow |f(z_0 + \rho e^{i\theta}) - f(z_0)| < \varepsilon \qquad ☺$$

So, $|②| \Rightarrow$

$$\left| \int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| \leq \int_{C_\rho} \frac{|f(z) - f(z_0)|}{|z - z_0|} dz$$

$$= \frac{1}{\rho} \int_{C_\rho} |f(z) - f(z_0)| dz$$

$$< \frac{1}{\rho} \varepsilon \cdot 2\pi\rho \text{ via } ☺$$

$$= 2\pi \varepsilon.$$

Send $\varepsilon \searrow 0 \Rightarrow ①'$.