LECTURE 23

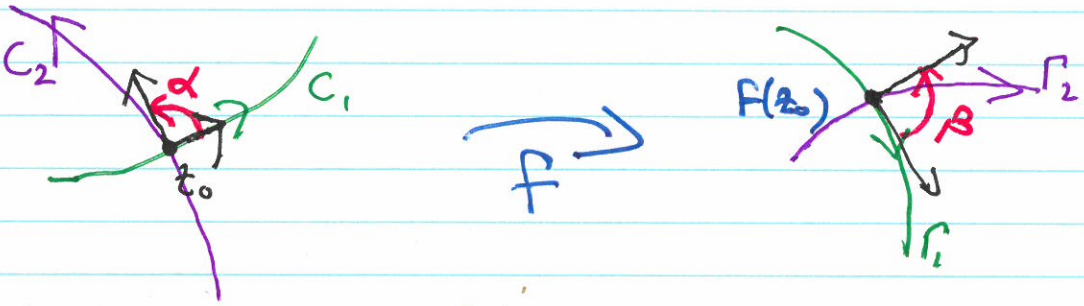
§112 (8 Ed §101) conformal maps.

- $f: z \mapsto w$
- f: analytic
- $f'(z_0) \neq 0$.

Then locally (near z_0), f preserves

- angle;
- orientation;
- shape.

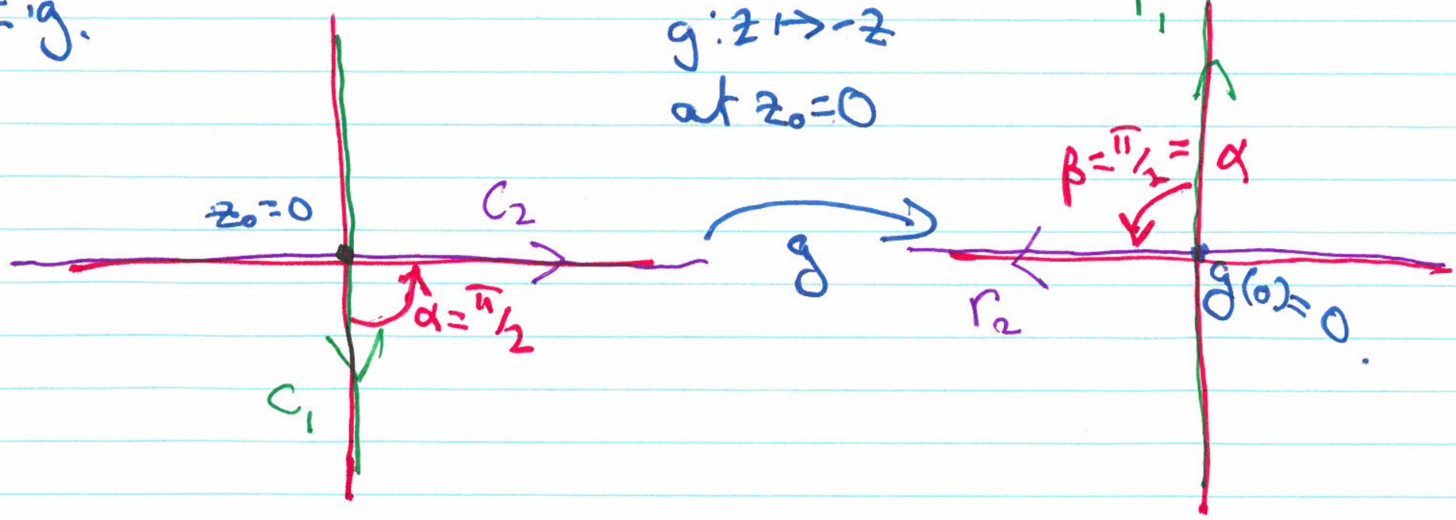
f is called conformal at z_0.



$\Gamma_1 = f(C_1)$, $\Gamma_2 = f(C_2)$

$\boxed{\text{Conformality} \Rightarrow \beta = \alpha.}$

E.g.

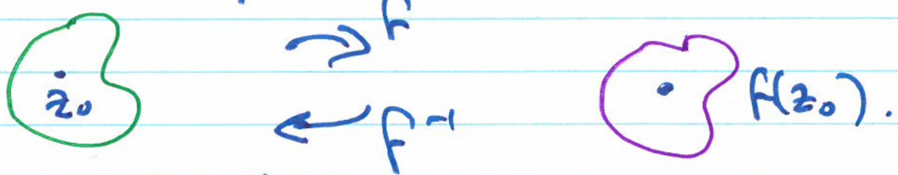


f analytic with $f'(z_0)=0$: z_0 is a critical pt. of f. Angle will not be preserved.

Can show: angle will be multiplied by m, where m is the smallest integer s.t. $f^{(m)}(z_0)\neq 0$.

RMK: If orientation is not necessarily preserved, but angle magnitude is, the map is called isogonal.

Important: conformality $\Rightarrow$ locally 1-1 & onto, in particular, f has a local inverse.



Inverse f^n th^m (MATH2400/2401) in $\mathbb{R}^2$ says $f:(x,y)\mapsto(u,v)$ sufficiently smooth is invertible if

$$\det(J_f) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0.$$

Jacobian (mx).

In our case, we know f is analytic, so u_x, u_y, v_x, v_y are all cts in a nbhd of $x_0+iy_0=z_0$, &

$$\det(J_f) = u_x \underbrace{v_y}_{=u_x} - \underbrace{u_y}_{=-v_x} v_x \quad \text{via C/R}$$

$$= u_x^2+v_x^2$$

$$=|u_x+iv_x|^2$$

$$=|f'(z_0)|^2>0. \qquad \square$$

Harmonic f^n's in $\mathbb{R}^n$

$\Omega \subseteq \mathbb{R}^n$. Look for $U : \Omega \to \mathbb{R}^n$

s.t. $\Delta U = 0$

Laplacian Laplace operator

(sometimes ∇^2)

Where $$\Delta U = \sum_{j=1}^{n} U_{jj} = \sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_j^2} = \sum_{j=1}^{n} D_{jj} U$$

in $\mathbb{R}^2$: $\Delta U = U_{xx} + U_{yy}$.

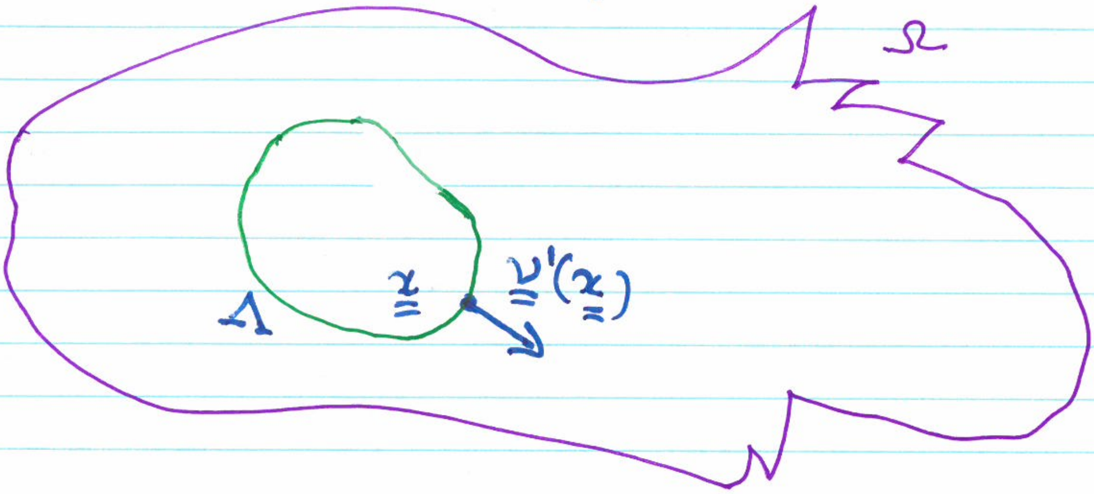
in $\mathbb{R}^3$: $\Delta U = U_{xx} + U_{yy} + U_{zz}$.

Models many physical situations in "steady state".

Motivation: $\Omega \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$.

Λ: "sufficiently smooth subdomain of Ω" with an exterior normal $\underline{\underline{\nu}}(\underline{\underline{x}})$ on $\partial\Lambda$, & exterior unit normal $\underline{\underline{\nu}}'(\underline{\underline{x}})$.

"$=$" denotes a vector quantity.



U = density of something "in equilibrium".

$\underline{\underline{F}}$ = Flux density of U "in equilibrium" in Ω.

$\int_{\partial\Lambda} \underline{\underline{F}} \cdot \underline{\underline{\nu}}' \, dS = 0$. ($dS$ = surface measure on $\partial\Lambda$).

Gauß $\Rightarrow \int_{\Lambda} \operatorname{div} \underline{\underline{F}} \, d\underline{\underline{x}} = 0$.

$d\underline{\underline{x}} = dx\,dy$ in $\mathbb{R}^2$

$= dx\,dy\,dz$ in $\mathbb{R}^3$.

Since Δ is (essentially) arbitrary, there holds: $\operatorname{div} \underline{F} = 0$ in Ω

i.e. $\sum_{j=1}^{n} \partial_j \underline{F} = 0$ in Ω (*)

In many physical situations,

$\underline{F} = c \nabla U$, with typically negative.

(*) $\Rightarrow$ $c(\operatorname{div} \nabla U) = 0$

i.e., $\Delta U = 0$.

Can also study $\frac{\partial U}{\partial t} = \alpha \Delta U$ (*)

- If U is the concentration of a chemical, (*) is Fick's Law of chemical diffusion;
- If U is temperature, (*) is Fourier's Law of heat conduction;
- If U is the electric potential, (*) is Ohm's law of electrical conduction.