LECTURE 23

A key step in the proof of Cauchy-Goursat is the M-ℓ estimate:

Suppose f is cts on a contour C given by $z(t)$ $a \le t \le b$.

Then $f \circ z : [a,b] \to \mathbb{C}$ is cts;
$|f \circ z| : [a,b] \to \mathbb{R}$, so extreme value th^{m} in $\mathbb{R}$ from MATH2400/2401/MATH1071

$\Rightarrow$ $\exists M$ s.t. $|f(z)| \le M$ $\forall z \in C$.

So
$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\, dt\right|$$
$$\le \int_a^b |f(z(t))| \cdot |z'(t)|\, dt.$$
$$\le M \int_a^b |z'(t)|\, dt$$
$$= M\ell$$

where $\ell = \ell(C) =$ length of C.

This is the M-ℓ estimate: $\boxed{\int_C f(z)\,dz \le M\ell}$

Extensions of Cauchy's Integral formula.

Setting: f is analytic in and on C_1, where C_1 is a simple closed contour in $\mathbb{C}$, traversed in the +ve sense, $z_0 \in \text{Int } C_1$.

Cauchy (Lec 22) $\Rightarrow$ $f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz$.

Thm. §55 (§52 8Ed)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_1} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \text{ for } \quad (1)$$

for $n \geq 1$

nth derivative of f at z_0.

works for $n = 0$

$0! = 1$, $f^{(0)} = f$.

Pf: see ex 9 §57 (8Ed Ex 9 §52).

THM ⊛ If f is analytic at z_0, then its derivatives of all orders exist & are analytic at z_0.

Pf: f is analytic at $z_0 \Rightarrow f$ is analytic on $B_\varepsilon(z_0)$ for some $\varepsilon > 0$. Then for $C_1 = \{z_0 + \frac{\varepsilon}{2}e^{i\theta}, 0 \leq \theta \leq 2\pi\}$, f is analytic on C_1 & in Int C_1, so

(1) $\Rightarrow f''(z) = \frac{1}{\pi i}\int_{C_1} \frac{f(s)}{(s-z)^3}\,ds \quad \forall z \in \text{Int } C_1$.

$\Rightarrow f''$ exists on Int C_1 $\Rightarrow f'$ is analytic on Int C_1 & at z_0 in particular.

Reapply to f' to show f'' is analytic, etc. $\square$

cf. the situation in $\mathbb{R}$: e.g., $f(x) = |x|^3$: f, f', f'' are all cts on $\mathbb{R}$, but $f'''(0)$ does not exist.

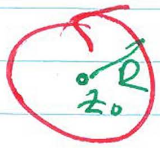
RMK: For $f = u + iv$: f analytic at $z_0 = x_0 + iy_0$ $\Rightarrow$ partials of all orders exist & are cts at (x_0, y_0).

THM (Morera) Let f be cts on a domain $\Omega \subseteq \mathbb{C}$. If $\int_C f = 0$ $\forall$ closed contours lying in Ω. Then f is analytic in Ω.

Pf: By Th^m §48 (Lec 20), f has a primitive on Ω, say F. But then $F' = f$ exists & is cts on Ω by conditions of th^m. But then $F' = f$ exists & is cts on on Ω by conditions of Th^m, so F is analytic. So by Th^m (*), $f = F'$ is also analytic on Ω. $\square$

A number of very important/nice/useful results follow from (1).

Let f be analytic in & on $C_R(z_0) = \{z_0 + Re^{i\theta} \; 0 \le \theta \le 2\pi\}$



Then I. $\boxed{|f^{(n)}(z_0)| \le \frac{n!\, M_R}{R^n}}$ (2)

where $M_R = \max_{z \in C_R} |f(z)|$.

<u>Pf:</u> note that M_R is well defined & finite by Extreme Value Th^m.

$$|f^{(n)}(z_0)| \overset{(1)}{=} \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\le \frac{n!}{2\pi} \int_{C_R} \frac{\underbrace{|f(z)|}_{\le M_R}}{\underbrace{|z-z_0|^{n+1}}_{= R^{n+1} \text{ on } C_R}} dz$$

$$\le \frac{n!\, M_R}{2\pi R^{n+1}} \int_{C_R} dz$$

$$= \frac{n!\, M_R}{\cancel{2\pi R} \cdot R^n} \cdot \cancel{2\pi R}$$

$$= \frac{n!\, M_R}{R^n}$$

$\square$

<u>Rmk:</u> leads to e.g. Bieberbach conjecture.

II. Liouville's Th^m.

If: $f:\mathbb{C}\to\mathbb{C}$ is bdd & entire, then f is constant.

Pf: Suppose that $|f|<M$ on f, f is entire, $z_0\in\mathbb{C}$.

Apply (2) for $n=1$ on $C_R=\{z_0+Re^{i\theta},\ 0\le\theta\le 2\pi\}$

$$|f'(z_0)|\le\frac{1!\,M}{R}=M/R \quad (**)$$

Letting $R\to\infty$ & noting LHS of (**) is independent of R, we see there must hold $f'(z_0)=0$. Since z_0 was arbitrary, f must be constant. $\square$

IMPORTANT: entire is crucial here. $\exists$ non-constant analytic f^{n}s on "large" domains in $\mathbb{C}$.

E.g: on the UHP (upper half plane) $\{z: \operatorname{Im} z>0\}$ consider $z\mapsto e^{iz}$

For $z=x+iy$ in UHP $y>0$

$$|e^{iz}|=|e^{i(x+iy)}|=|e^{ix-y}|=e^{-y}<1$$

So e^{iz} is bdd on UHP.

III. Fund. th^m of algebra (Pf: Applied sessions, wk 9).

Next: §112 conformal maps.