LECTURE 24

Harmonic f^n's are sol^n's of Laplace's eq^n:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

Examples of harmonic f^n's:

* On $\mathbb{R}^2 \setminus \{(0,0)\}$:

$$f(x,y) = \ln(\|(x,y)\|) = \ln\left(\sqrt{x^2+y^2}\right) \text{ is harmonic.}$$

* On $\mathbb{R}^n \setminus \{\underline{0}\}$ $\quad n \geq 3$

$$f(\underline{x}) = \frac{1}{\|\underline{x}\|^{n-2}} = \frac{1}{(x_1^2+\cdots+x_n^2)^{n-2/2}} \text{ is harmonic.}$$

* On $\mathbb{R}$: $f(x) = ax+b$ $\quad a,b \in \mathbb{R}$.

Focus on $\mathbb{R}^2$.

A f^n $u : \underset{\subseteq \mathbb{R}^2}{\Omega} \to \mathbb{R}$ is harmonic if:

* u & its derivatives of first order (i.e., u_x & u_y) exist & are cts on Ω;
* u_{xx} & u_{yy} are cts on Ω; &

$u_{xx} + u_{yy} = 0$, i.e., $\Delta u = 0$ in Ω.

THM: If $f(z) = u(x,y) + iv(x,y)$ is analytic in $\Omega \subseteq \mathbb{C}$, then u & v are harmonic.

PF: recall §55: f is analytic $\Rightarrow$ u & v have cts partials of all orders & C/R hold in Ω, i.e.,

$$u_x = v_y \;\&\; u_y = -v_x \qquad (1)$$

∂_x of (1) $\Rightarrow$ $u_{xx} = v_{yx}$ & $u_{yx} = -v_{xx}$ (2)

∂_y of (1) $\Rightarrow$ $u_{xy} = v_{yy}$ & $u_{yy} = -v_{xy}$ (3)

Since all partials of all orders are cts,

$u_{xy} = u_{yx}$ & $v_{xy} = v_{yx}$ (Clairaut's thm)

So (2) & (3) $\Rightarrow$ Δu & $\Delta v = 0$ Ω

Def: if u & v are harmonic & satisfy C/R, then v is called a (not "the") harmonic conjugate of u.

Thm: $f = u + iv$ is analytic in $\Omega \iff$ v is a harmonic conjugate of u.

PF: ($\Rightarrow$) done in previous thm.

($\Leftarrow$) v is a harmonic conjugate of u says u & v are both harmonic, so u, v, u_x, u_y, v_x, v_y exist & are cts; & C/R hold throughout $\Omega \Rightarrow$ $f = u + iv$ is analytic in Ω. Ω

Finding harmonic conjugates.

Firstly note:

Suppose v & w are harmonic conjugates of u.

$\Rightarrow u+iw$ & $u+iv$ are both analytic.

C/R I $\Rightarrow u_x = v_y = w_y$ (*)

Integrate (*) w.r.t. $y \Rightarrow v = w + \phi(x)$ (4)

C/R II $\Rightarrow u_y = -v_x = -w_x$ ☺

Integrate ☺ w.r.t. $x \Rightarrow v = w + \psi(y)$ (5)

Compare (4) & (5) $\Rightarrow \phi(x) = \psi(y)$, which must be a constant, i.e., $\in \mathbb{R}$

$$\boxed{v = w + c}$$

Use a similar procedure to find the harm. conj. of a given f: u.

E.g., $u(x,y) = y^3 - 3x^2y$ on $\mathbb{R}^2$: find a harm. conj. of u.

Sol'n: u is a polynomial f'n of x & y, so is cts & has cts partials of all orders.

Further: $u_{xx} + u_{yy} = 0$ ☆

Let v be a harm conj. of u.

$C/R_I \Rightarrow u_x = v_y$,

i.e., $v_y = -6xy$.

Integrate w.r.t. y $\Rightarrow v = -3xy^2 + \phi(x)$ (6)

$C/R_{II} \Rightarrow u_y = -v_x$

$3y^2 - 3x^2 = 3y^2 - \phi'(x)$

$\Rightarrow \phi'(x) = 3x^2$

$\Rightarrow \phi(x) = x^3 + C \qquad C \in \mathbb{R}$.

So $(C=0) \Rightarrow v = -3xy^2 + x^3$ is a harm conj of u.

Note $u + iv = (y^3 - 3x^2y) + i(-3xy^2 + x^3)$,

so for $f(z) = iz^3$, $u = Re(f)$ & $v = Im(f)$.

RMK: v is a harm. conj. of u $\not\Rightarrow$ u is a harm conj. of v.

E.g. $u = x^2 - y^2$, $v = 2xy$.

$u + iv = (x+iy)^2 = z^2$, which is entire.

$\Rightarrow v$ is a harm. conj. of u.

BUT, if u were a harm. conj. of v, then $g = v + iu$ would be analytic

check with C/R: g is nowhere analytic.

RMK: Suppose u is harmonic on a simply connected domain Ω.

Then (§115 8EA §104)

u has a harm conj. on Ω.

Next physical problems.