LECTURE 24

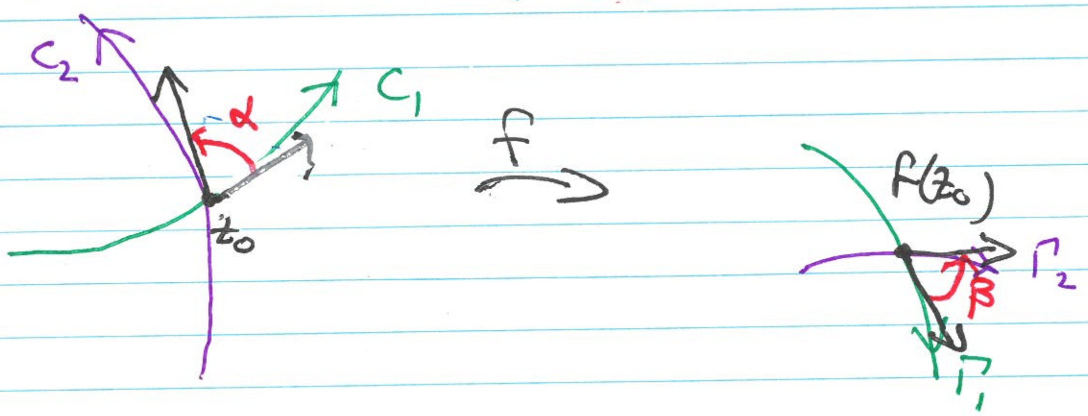
§112 (8 Ed §101) Conformal Maps

* $f: z \mapsto w$;
* f analytic ;
* $f'(z_0) \neq 0$.

Then locally (near z_0) f preserves

* angle ;
* orientation ;
* shape .

f is called conformal at z_0.



$\Gamma_1 = f(C_1)$, $\Gamma_2 = f(C_2)$.

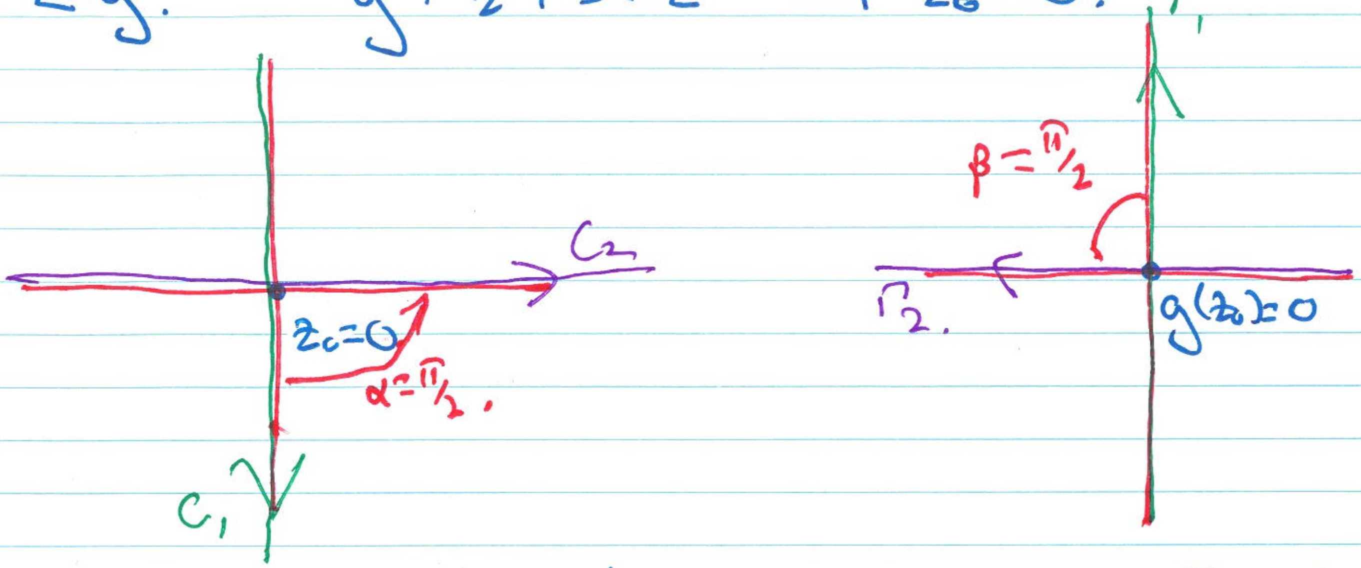
$$\boxed{\text{Conformality} \Rightarrow \beta = \alpha}$$

Roughly : $w = f(z)$, $z = z(t)$:

$$w' = f'(z(t)) \cdot z'(t).$$

$$\Rightarrow \arg w' = \arg f' + \arg z'$$

E.g: $g: z \mapsto -z$ at $z_0 = 0$.

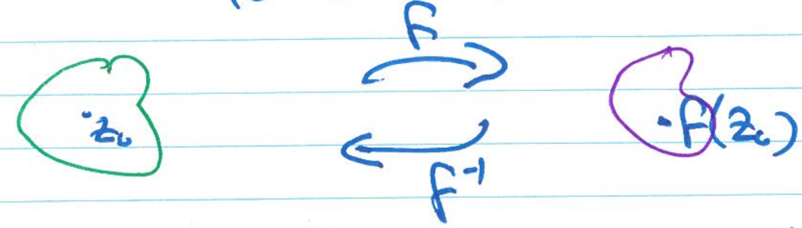


Analytic f: with $f'(z_0) = 0$: z_0 is a critical value of f. Angle will not be preserved.
Can show: angle will be multiplied by m, where m is the smallest integer s.t.

$$\boxed{f^{(m)}(z_0) \neq 0.}$$

RMK: If orientation is not necessarily preserved, but angle magnitude is, the map is called isogonal, e.g., $z \mapsto \bar{z}$.

Conformality $\Rightarrow$ locally 1-1 & onto, i.e., f has a local inverse.



Inverse f? th^m (MATH2400/2401) in $\mathbb{R}^2$ says: $f : (x,y) \mapsto (u,v)$ sufficiently smooth is invertible if

$$\det(J_f) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0.$$

Jacobian

In our case: f is analytic $\Rightarrow u_x, u_y, v_x, v_y$ are all cts in a nbhd of $z_0 = x_0 + iy_0$,

$$\& \quad \det(J_f) = \underset{\underset{u_x}{\shortparallel}}{u_x v_y} - \underset{\underset{-v_x}{\shortparallel}}{u_y} v_x$$

$$= u_x^2 + v_x^2$$

$$= |u_x + i v_x|^2$$

$$= |f'|^2 \neq 0 \quad \text{at } z_0. \qquad \square$$

Harmonic f^n's in $\mathbb{R}^n$

$\Omega \subseteq \mathbb{R}^n$: look $U : \Omega \to \mathbb{R}$

s.t. $\Delta U = 0$

Laplacian or Laplace operator.

(sometimes ∇^2)

del or nablaa

where $\Delta U = \sum_{j=1}^{n} U_{jj} = \sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_j^2}$

in $\mathbb{R}^2$: $\Delta U = U_{xx} + U_{yy}$;

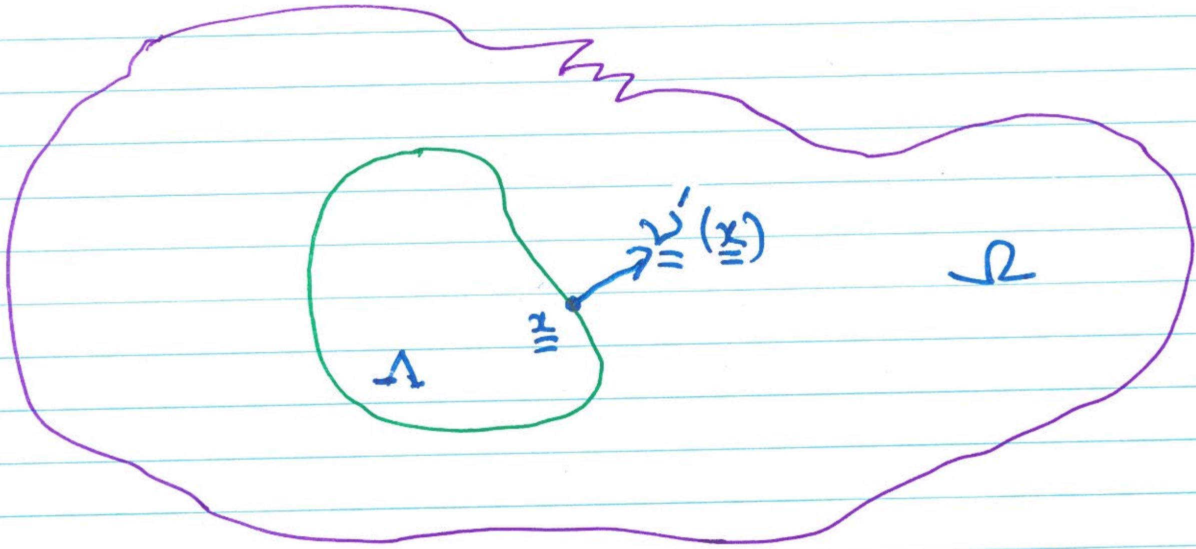
in $\mathbb{R}^3$: $\Delta U = U_{xx} + U_{yy} + U_{zz}$.

Models many physical situations in "steady state".

Motivation : $\Omega \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$.

Λ : "sufficiently smooth" subdomain of Ω with an exterior normal $\underline{\underline{\nu}}(\underline{\underline{x}})$ on $\partial\Lambda$, & external unit normal $\underline{\underline{\nu}}'(\underline{\underline{x}})$.

"$=$" denotes a vector quantity.



$\mathcal{U}$ density of something "in equilibrium" in Ω

F = flux density of $\mathcal{U}$ "in equilibrium" in Ω.

$\int_{\partial\Lambda} \underline{\underline{F}} \cdot \underline{\underline{\nu}}' \, dS = 0$ $\quad dS$ = surface measure on $\partial\Lambda$

Gauss $\Rightarrow \int_{\Lambda} \operatorname{div} \underline{\underline{F}} \, d\underline{\underline{x}} = 0$

$d\underline{\underline{x}} = dx\,dy$ in 2D

$= dx\,dy\,dz$ in 3D.

Since Δ is (essentially) arbitrary, there must hold:

$$\operatorname{div} \underline{\underline{F}} = 0 \quad \text{in } \Omega,$$

i.e., $$\sum_{j=1}^{n} \partial_j \underline{\underline{F}}_j = 0 \quad \text{in } \Omega. \quad (*)$$

In many physical situations,

$$\underline{\underline{F}} = c \nabla U, \quad \text{with } c \text{ typically -ve}$$

$$(*) \Rightarrow \quad c(\operatorname{div} \nabla U) = 0$$

i.e., $$\Delta U = 0.$$

Can also study: $$\frac{\partial U}{\partial t} = a \Delta U \quad (*)$$

- If U is the concentration of a chemical, $(*)$ is Fick's law of chemical diffusion;
- If U is temperature, $(*)$ is Fourier's law of heat conduction;
- If U is electric potential, $(*)$ is Ohm's law of electrical conduction.