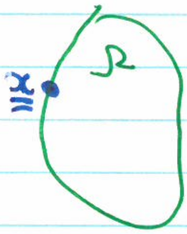
LECTURE 25

§116 (8Ed §115) PHYSICAL PROBLEMS

Physical configurations are often modelled by sol^{n}s of partial differential equations (PDE). Generally, interested in solving PDE subject to given initial & or boundary conditions.

e.g.

$$\text{(D)} \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi & \text{on } \partial\Omega \ (*) \end{cases}$$



$u : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$

(*) says: $u(\underline{x}) = \varphi(\underline{x})$ for $\underline{x} \in \partial\Omega$.

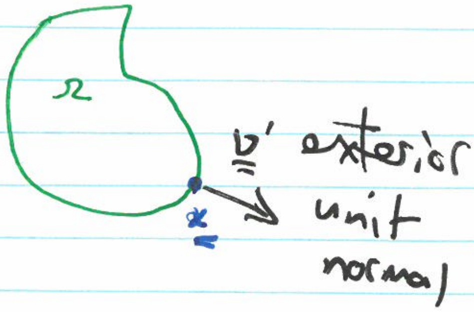
$\varphi : \partial\Omega \to \mathbb{R}$, Ω, φ are known/given.

u is the unknown f^n we are looking for.

(D) is called the Dirichlet Problem for Laplace's equation, a.k.a. boundary problem of the first kind.

One way to "solve" (D) is to find u that minimises $\int_\Omega |\nabla u|^2 dx$ subject to $u|_{\partial\Omega} = \varphi$.

(rmk: show $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega |\nabla(u+\varepsilon\Phi)|^2 d\underline{x} = 0$), Also important: boundary cond^n's of the second kind, a.k.a. Neumann bdy conditions.

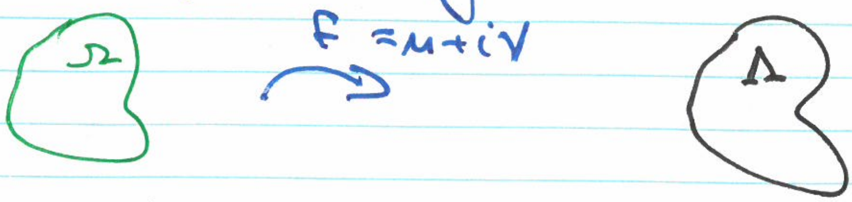


$$(N)\ \begin{cases} \Delta u = 0 & \text{in } \Omega. \\ \dfrac{\partial u}{\partial \underline{\underline{\nu}}'} = \psi & \text{on } \partial\Omega \end{cases}$$

$$\frac{\partial u}{\partial \underline{\underline{\nu}}'} = \nabla u(\underline{\underline{x}}) \cdot \nu'(\underline{\underline{x}})$$

In practice, often have homogeneous Neumann b.c., i.e., $\psi = 0$ (no-slip conditions).

Transformations of harmonic f^{n}s.



$z = x+iy$ $\qquad\qquad$ $w = u+iv$.

Th^m: if f is conformal & h is harmonic in Λ, then H is harmonic in Ω, where
$$H(x,y) = h(u(x,y), v(x,y)).$$

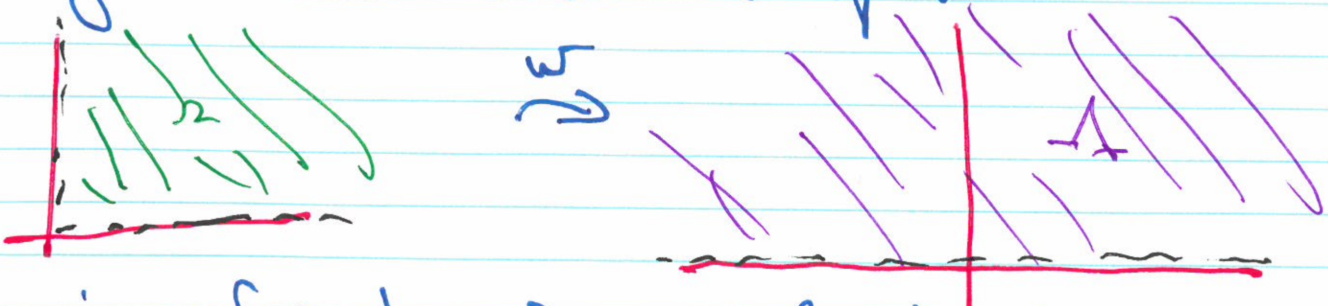
Pf: messy in general, relatively straightforward if Λ is simply connected (§115, 8Ed §104)

E.g. $h(u,v) = e^{-v}\sin u$ is harmonic in the UHP (upper half plane).

To see that, note that e^{-v} is C^∞, as is $\sin u$.
$\Rightarrow h$ is C^∞: in particular, h & all partials exist & are cts on $\mathbb{R}^2$.

$$\left.\begin{aligned} h_u &= e^{-v}\cos u & h_{uu} &= -e^{-v}\sin u; \\ h_v &= -e^{-v}\sin u & h_{vv} &= e^{-v}\sin u. \end{aligned}\right\} \Rightarrow \Delta h = 0.$$

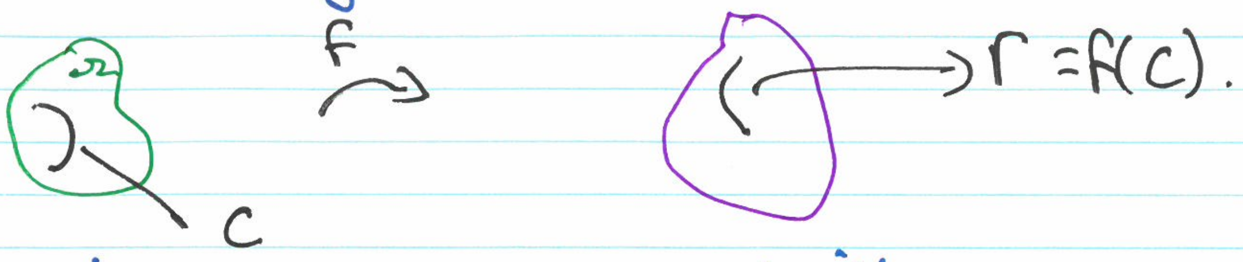
Define $w = z^2$ on Ω = 1st quadrant.



w is conformal on Ω, $u = x^2 - y^2$, $v = 2xy$.

Th^m $\Rightarrow H(x,y) = e^{-2xy}\sin(x^2-y^2)$ is harmonic on Ω.

Add in bdy cond's as well:



$z = x+iy$ $\qquad\qquad$ $w = u+iv$

f conformal

C smooth (C^∞) curve in Ω.

(can take C in $\partial\Omega$ with a bit more care).

$$H(x,y) = h(u(x,y)\, v(x,y))$$

① Dirichlet b.c. on Γ, i.e.,

$h(u,v) = \varphi$ on Γ: then

$$\boxed{H(x,y) = \varphi(u(x,y), v(x,y)) \text{ on } C}$$

② If we have homogeneous Neumann b.c. on Γ, i.e., $\frac{\partial h}{\partial \underline{n}} = 0$ for $\underline{n}$ normal to Γ, then

$$\boxed{\frac{\partial H}{\partial \underline{N}} = 0 \text{ for } \underline{N} \text{ normal to } C}$$