LECTURE 25

Harmonic f^{n}s are sol^{n}s of Laplace's eq^n:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

Examples:

On $\mathbb{R}^2 \setminus \{(0,0)\}$:

* $f(x,y) = \ln(\|(x,y)\|) = \ln\left(\sqrt{x^2+y^2}\right)$ is harmonic. ☆

* On $\mathbb{R}^n \setminus \{\underline{0}\}$, $n \geq 3$

  $f(\underline{x}) = 1/\|\underline{x}\|^{n-2} = 1/(x_1^2 + \dots + x_n^2)^{\frac{n-2}{2}}$ is harmonic. ☆

* On $\mathbb{R}$: $f(x) = ax+b$ $\quad a,b \in \mathbb{R}$.

(Our) precise def^n of a harmonic f^n:

$\Omega \subseteq \mathbb{R}^2$. $u : \Omega \to \mathbb{R}$ is harmonic if:

* u & its derivatives of first order (i.e., u_x & u_y) exist & are cts on Ω;
* u_{xx} & u_{yy} are cts on Ω; &
* $u_{xx} + u_{yy} = 0$, i.e., $\Delta u = 0$ in Ω.

THM: If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in open $\Omega \subseteq \mathbb{C}$, then u & v are harmonic.
$\quad\quad\;\; \overset{\shortparallel}{x+iy}$

Pf: recall §55: f is analytic $\Rightarrow$ u & v have cts partials of all orders & C/R hold in Ω, i.e.,

$$u_x = v_y \quad \& \quad u_y = -v_x. \quad (1)$$

∂_x of (1) $\Rightarrow u_{xx} = v_{yx}$ & $u_{yx} = -v_{xx}$ (2)

∂_y of (1) $\Rightarrow u_{xy} = v_{yy}$ & $u_{yy} = -v_{xy}$ (3)

Since all partials of all orders are cts,

$u_{xy} = u_{yx}$ & $v_{xy} = v_{yx}$ (Clairaut's Th$^{\underline{m}}$)

So (2) & (3) $\Rightarrow \Delta u = 0$ & $\Delta v = 0$. Ω

Def: If u & v are harmonic & satisfy C/R, then v is called a harmonic conjugate of u.
not "the"

THM for Ω open: $f = u + iv$ is analytic in $\Omega \iff$ v is a harmonic conjugate of u.

Pf: ($\Rightarrow$) done in previous Th$^{\underline{m}}$.

($\Leftarrow$) v is a harm. conj of $u \Rightarrow u$ & v are both harmonic, so v, u, u_x, u_y, v_x, v_y exist & are cts in Ω. Further, C/R hold throughout $\Omega \Rightarrow$

$f = u + iv$ is analytic. Ω

Finding harmonic conjugates.

Firstly, note:

Suppose v & w are harmonic conjugates of u.

$\Rightarrow$ $u+iv$ & $u+iw$ are both analytic.

C/R$_{\text{I}}$ $\quad u_x = \underbrace{v_y = w_y}_{*}$.

Integrate $*$ w.r.t. y $\Rightarrow$ $v = w + \phi(x)$ $\quad$ (4)

for some f^n ϕ of x only.

C/R$_{\text{II}}$ $\quad \Rightarrow u_y = \underbrace{-v_x = -w_x}_{☺}$.

Integrate w.r.t. x $\quad \Rightarrow \quad v = w + \psi(y)$ $\quad$ (5)

Compare (4) & (5) $\Rightarrow$ $\phi(x) = \psi(y)$, which hence must be a constant $\in \mathbb{R}$, i.e.,

$$\boxed{v = w + c.}$$

Use a similar procedure/argument to find the harm. conj. of a given harm. f^n.

E.g: $u(x,y) = y^3 - 3x^2 y$ on $\mathbb{R}^2$: find a harm. conj of u.

Sol^n: u is a polynomial f^n of x & y, so has cts partials of all orders.

Further $u_{xx} + u_{yy} = 0$ ☆

Let v be a harm conj. of u.

$CR_I \Rightarrow u_x = v_y$

i.e., $v_y = -6xy$.

Integrate this w.r.t. $y \Rightarrow v = -3xy^2 + \phi(x)$ (6)

$CR_{II} \Rightarrow u_y = -v_x$

(6) $\Rightarrow \cancel{3y^2} - 3x^2 = \cancel{3y^2} - \phi'(x)$.

$\Rightarrow \phi'(x) = 3x^2$

$\Rightarrow \phi(x) = x^3 + c \qquad c \in \mathbb{R}$

So $(c=0) \Rightarrow v = -3xy^2 + x^3$ is a harm conj. of u. $\square$

Note: for $f(z) = iz^3$, $u = \mathrm{Re}(f)$ & $v = \mathrm{Im}(f)$.