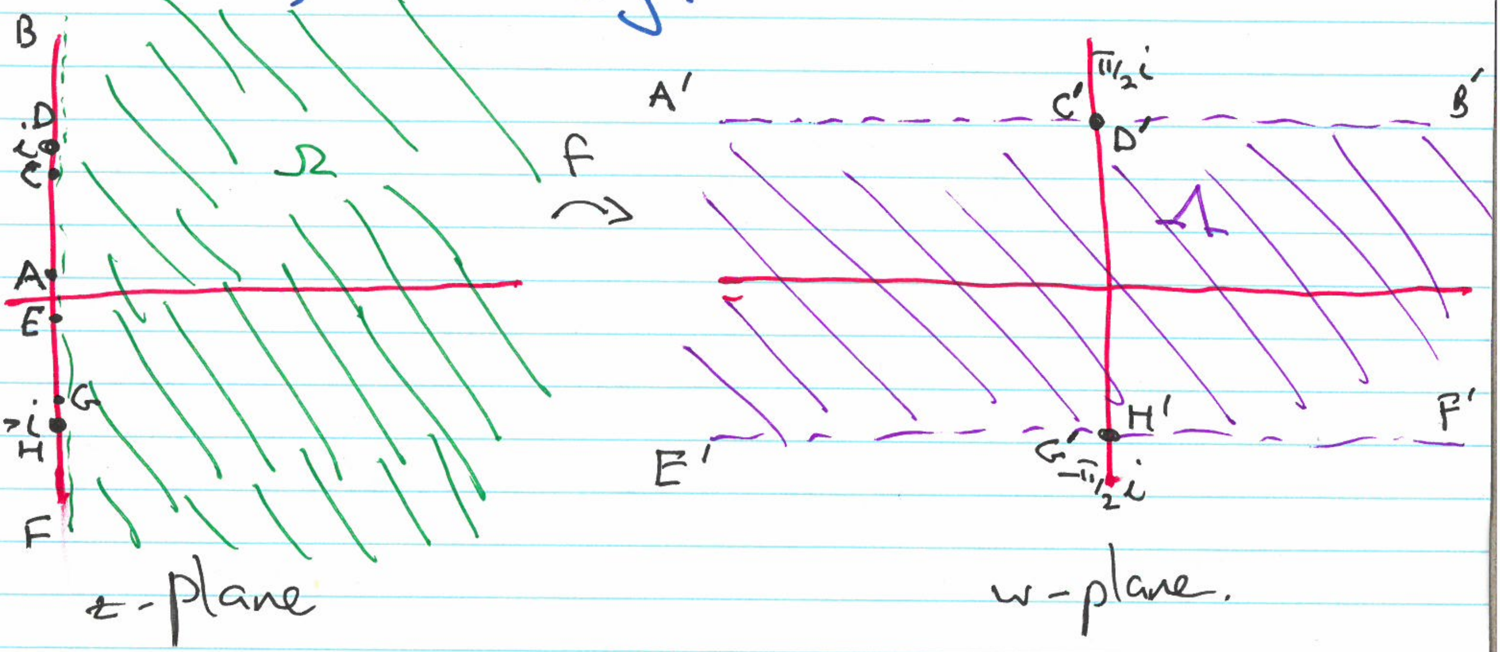
LECTURE 26

Example: in $\mathbb{C}$ (w-plane), the function $h(u,v) = v = \operatorname{Im} w$ is harmonic, & in particular is harmonic on the horizontal strip Λ: $-\pi/2 < v < \pi/2$.

Claim: $f: z \mapsto \operatorname{Log} z$ maps Ω = Right Half Plane onto Λ, conformally.



$$z \mapsto \operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

A is of the form $0 + \delta i$, $0 < \delta \ll 1$.

$$A' = f(A) = \operatorname{Log} A = \ln|A| + i \operatorname{Arg} A$$
$$= \underbrace{\ln \delta}_{\text{large -ve}} + i \underbrace{\operatorname{Arg}(\delta i)}_{\pi/2}$$

Conformality: $f'(z) = 1/z$ doesn't vanish on Ω & f is analytic.

Note $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$
$$= \ln\sqrt{x^2+y^2} + i \arctan y/x \text{ in } \Omega$$

$\Rightarrow H(x,y) = \arctan y/x$ is harmonic in Ω.

P.S. on conformal mappings: scale factor

$f: z \mapsto w$ conformal (i.e., analytic, $f'(z_0) \neq 0$).

For z near z_0, $z \neq z_0$, $f'(z_0) \neq 0$.

$$\frac{|f(z) - f(z_0)|}{|z - z_0|} \approx |f'(z_0)|$$

$\Rightarrow |f(z) - f(z_0)| \approx |z - z_0| \, |f'(z_0)|$.

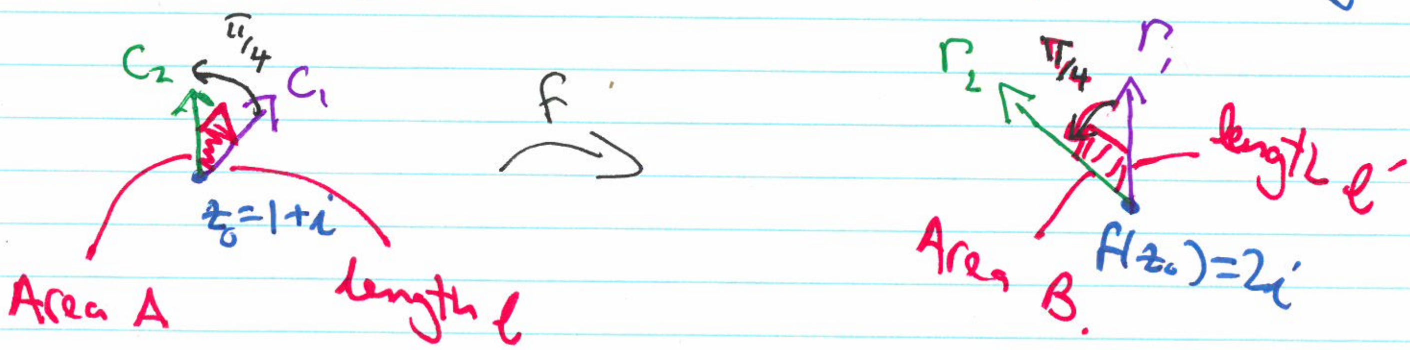
$\Rightarrow |f'(z_0)|$ is the scaling factor or dilation factor

stretching if $|f'(z_0)| > 1$,

shrinking if $|f'(z_0)| < 1$.

E.g. $f(z) = z^2$ at $1+i$.

$z = x+iy$, $w = u+iv$ $\quad u = x^2 - y^2$, $v = 2xy$.



scaling factor $= |f'(z_0)| = 2|z_0| = 2\sqrt{2}$.

$\Rightarrow \ell' \approx 2\sqrt{2}\,\ell$.

$\Rightarrow B \approx (2\sqrt{2})^2 A = 8A$.

§60 (8Ed §55) Sequences & Series

(cf. the situation in $\mathbb{R}$).

Formally, a sequence is a f^n: $\mathbb{N} \to \mathbb{C}$ (or $\mathbb{N}_0 \to \mathbb{C}$) $n \mapsto z_n$: write $\{z_n\}$.

Limit of a seq.:

$\lim_{n\to\infty} z_n = z \iff \{z_n\}$ converges to z

$\iff$ Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow$ $|z - z_n| < \varepsilon$.

Formally, a series $\sum_{n=0}^{\infty} z_n$ $z_n \in \mathbb{C}$ converges as a series $\iff$ the associated sequence of partial sums $\{S_n\}$ converges as a sequence.

Here: $S_n = \sum_{k=0}^{n} z_k$.

Typical q^n: Does $\sum z_n$ converge?

Test for NO: nth term test.

$\sum z_n$ converges $\Rightarrow z_n \to 0$.

$\not\Leftarrow$ e.g. $\sum(\frac{1}{n} + 0i)$ diverges.

RMK: $\{z_n\}$ is bounded says $\boxed{\exists M \text{ s.t. } |z_n| < M \ \forall n.}$

convergent $\Rightarrow$ bounded.

$\not\Leftarrow$ e.g. $\{(-1)^n + 0i\}$.

Def^n: $\sum z_n$ converges absolutely says $\sum |z_n|$ converges.

Abs. convergence $\Rightarrow$ convergence

$\not\Leftarrow$ e.g. $\sum \left(\frac{(-1)^n}{n} + 0i\right)$

Given $\sum_{n=0}^{\infty} z_n$, $S_n = \sum_{k=0}^{n} z_k$ as above,

& write $\rho_n = \sum_{k=n+1}^{\infty} z_k$ (ρ is the tail/remainder)

Then $S_n \to S \Leftrightarrow \sum z_n = S$ by def^n:

THM: $S_n \to S \Leftrightarrow \rho_n \to 0$. ☆

Applic: Claim: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z|<1$ (*)

geometric series

$= S$

Pf: $S_n = 1 + z + \cdots + z^n$ (a)

$zS_n = z + \cdots + z^n + z^{n+1}$ (b)

(a) − (b) $\Rightarrow$

$(1-z)S_n = 1 - z^{n+1} \Rightarrow S_n = \frac{1-z^{n+1}}{1-z}$.

$\rho_n = S - S_n = \frac{z^{n+1}}{1-z}$

Since $|z|<1$, $|\rho_n| \to 0$ as $n \to \infty$.

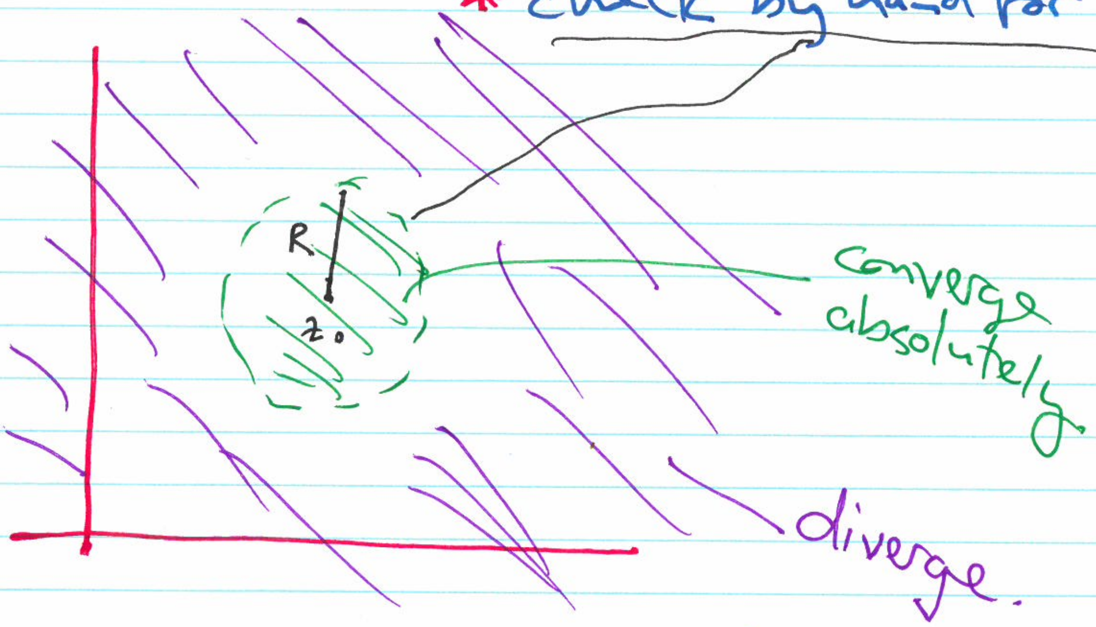
$\Rightarrow \rho_n \to 0$ as $n \to \infty$

$\Rightarrow S_n \to S$, showing (*)

RMK: As in $\mathbb{R}$ if $\sum a_n$ & $\sum b_n$ converge, so do $\sum(a_n \pm b_n)$, to $\sum a_n \pm \sum b_n$.
For $\lambda \in \mathbb{C}$, $\sum(\lambda a_n)$ cvrgs to $\lambda \sum a_n$.

POWER SERIES centred at z_0:
series o.t.f. $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Series will * converge absolutely for $|z-z_0|<R$, where R is the radius of convergence
* diverge for $|z-z_0|>R$
* check by hand for $|z-z_0|=R$.



* $R=0$: converges at $z=z_0$
$R=\infty$: converges on $\mathbb{C}$

"check by hand" c.f. situation in $\mathbb{R}$.

To determine radius of convergence R of $\sum a_n(z-z_0)^n$
Set $\lambda = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, then $R = 1/\lambda$

$\lambda = 0 \iff R=\infty$, $\lambda=\infty \iff R=0$.