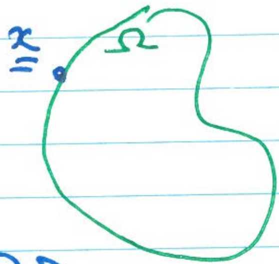
§116 (8Ed §115) PHYSICAL PROBLEMS

Physical configurations often modelled by solutions of partial differential equations (PDE).

Generally interested in solving PDE subject to given initial/boundary conditions.

e.g. (D) $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = \varphi & (*) \end{cases}$



(*) Say $u(\underline{\underline{x}}) = \varphi(\underline{\underline{x}})$ for $\underline{\underline{x}} \in \partial\Omega$.

$\varphi : \partial\Omega \to \mathbb{R}$, Ω, φ are known/given.

u is unknown.

(D) is called Dirichlet's problem for Laplace's equation, a.k.a. boundary value problem of the first kind.

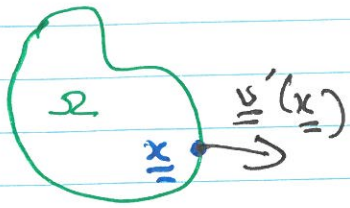
One way to "solve" (D) is to find u that minimises $\int_\Omega |\nabla u|^2 d\underline{\underline{x}}$ subject to $u\big|_{\partial\Omega} = \varphi$.

rmk: show that $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |\nabla(u+\varepsilon\phi)|^2 d\underline{\underline{x}}$

where ϕ is any perturbation with compact support in Ω.

└ set on which $\phi \neq 0$.

Also important: boundary conditions of the second kind, a.k.a. Neumann boundary conditions.



exterior unit normal

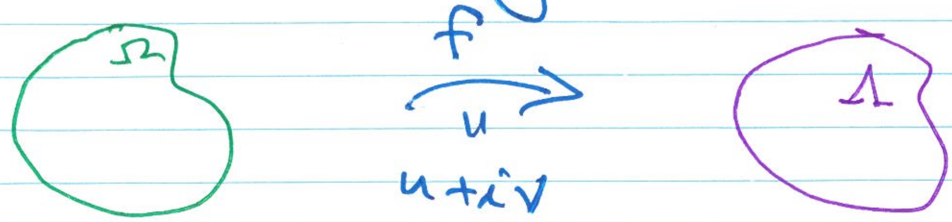
$$(N)\quad \begin{cases} \Delta u = 0 \\ \dfrac{\partial u}{\partial \underline{\underline{\nu}}'} = \psi \quad \text{on } \partial\Omega \end{cases}$$

$$= \nabla u(\underline{\underline{x}}) \cdot \underline{\underline{\nu}}'(\underline{\underline{x}}).$$

In practice, often have homogeneous Neumann boundary conditions,

i.e., $\psi = 0$ (no-slip condition).

Transformations of harmonic f^n's



$z = x+iy$ $\qquad\qquad$ $w = u+iv$.

Th^m: If f is conformal & h is harmonic in Λ, then H is harmonic in Ω, where $H(x,y) = h(u(x,y), v(x,y))$.

Pf: see §115, 8Ed §104.

E.g.: Note:

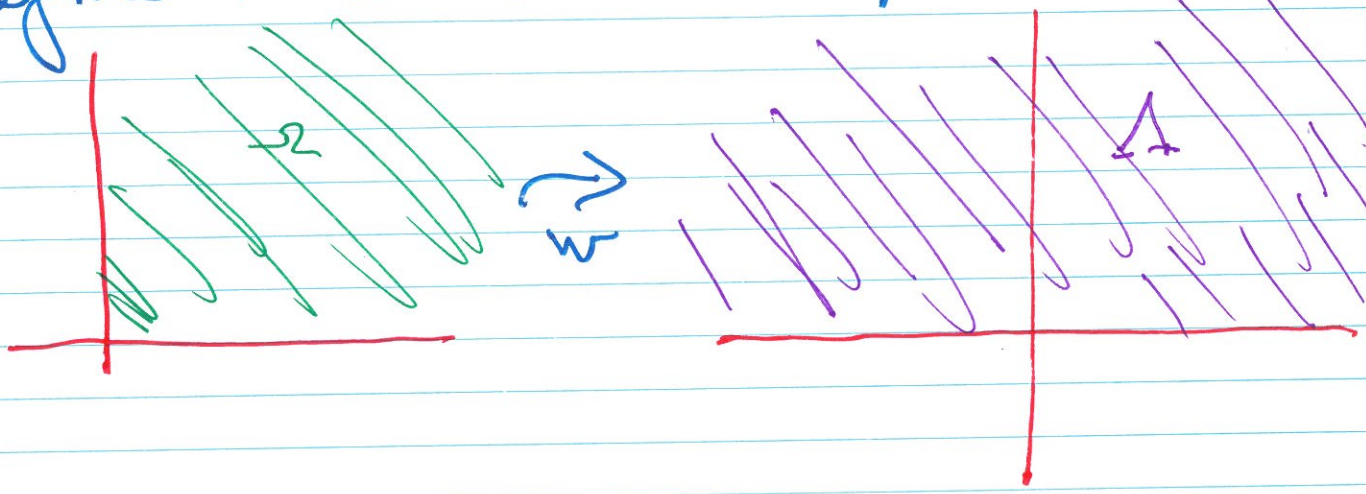
$h(u,v) = e^{-v}\sin u$ is harmonic in the UHP (upper half plane).

To see that: note that e^{-v} is C^∞, as is $\sin u$. $\Rightarrow h$ is C^∞: in particular, h & all partials exist & are cts on $\mathbb{R}^2$.

$$\left.\begin{array}{ll} h_u = e^{-v}\cos u & h_{uu} = -e^{-v}\sin u \\ h_v = -e^{-v}\sin u & h_{vv} = e^{-v}\sin u \end{array}\right\} \Rightarrow \Delta u = 0 \text{ on } \mathbb{R}^2$$

$(h(u,v) = e^{-v} \sin u)$.

Define $w = z^2$ on Ω = 1st quadrant.



w is conformal on Ω.

$u = x^2 - y^2, \quad v = 2xy.$

Th$^{\underline{m}}$: $H(x,y) = h(u(x,y), v(x,y))$

$= e^{-2xy} \sin(x^2 - y^2)$

is harmonic on Ω.