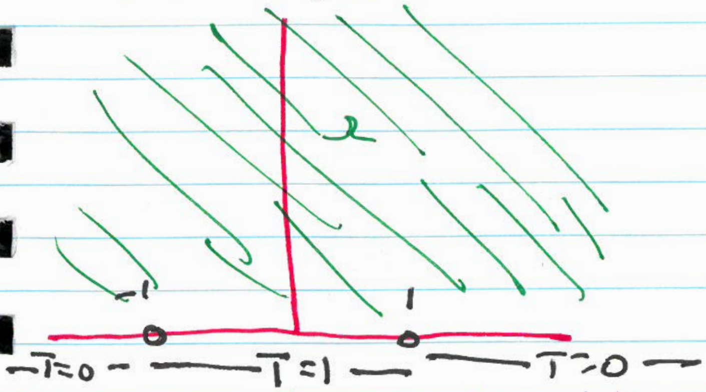
steady-state temperature in a half-plane.

(§119, 8Ed §107).



T = temperature.

want to find the steady state temperature distribution on Ω.

Heat conduction $\frac{\partial T}{\partial t} = \nabla \cdot (-k^2 \nabla T)$ t = time
$= -k^2 \Delta T$

steady state $\Rightarrow$ $\boxed{\Delta T = 0}$

So, want to solve

$$(D) \begin{cases} \Delta T = 0 & \text{in } \Omega \\ T(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}^{*} \end{cases}$$

* chosen to ensure bdy f^n defined at all points on bdy. Actual value at ±1 doesn't change result.

MATH3201 : solve numerically;
MATH3101 : Fourier series;
MATH3403 : Green's function;
MATH4403 : prove sol^n exists (unique).

physically: look for a bounded sol^n to (D), such that $\lim_{y \to \infty} T(x,y) = 0 \quad \forall x$.

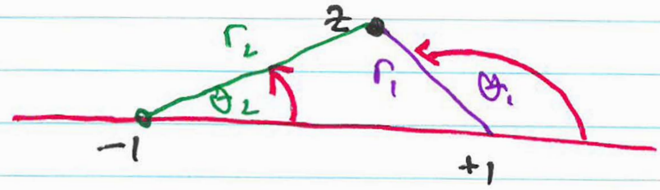
Define $\tilde{\Omega} = \{z : \operatorname{Im} z \geq 0, \ z \neq \pm 1\}$.

So, $\tilde{\Omega} = \Omega \cup \{\operatorname{Re} \text{ axis}\} \setminus \{\pm 1\}$.

On $\tilde{\Omega}$, define $\theta_1, \theta_2, r_1, r_2$ via

$$z-1 = r_1 \exp(i\theta_1)$$
$$z+1 = r_2 \exp(i\theta_2)$$

$r_1, r_2 > 0$, $0 \leq \theta_1, \theta_2 \leq \pi$.



Introduce the transformation

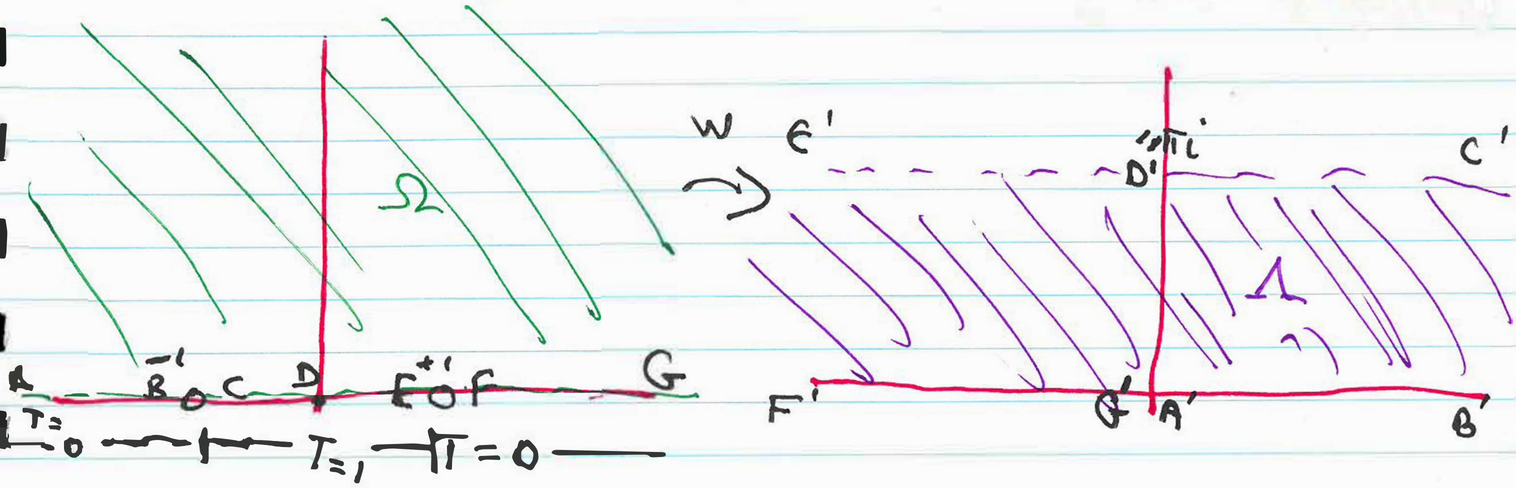
$$w = \operatorname{Log} \frac{z-1}{z+1}, \quad \text{where}$$

Log has a branch-cut on the -ve imaginary axis, so $-\pi/2 < \arg(\cdot) \leq 3\pi/2$.

So, $w = \operatorname{Log} \dfrac{r_1 \exp(i\theta_1)}{r_2 \exp(i\theta_2)}$

$$= \ln r_1/r_2 + i(\theta_1 - \theta_2).$$

Claim: w maps Ω onto the horizontal strip $0 < v < \pi$.



E.g. $A = -\Gamma + 0i \quad \Gamma \gg 1$

$r_1 = \Gamma + 1, \; r_2 = \Gamma - 1 \qquad \theta_1 = \theta_2 = \pi$

$$\Rightarrow A' = w(A) = \ln \frac{r_1}{r_2} + i(\theta_1 - \theta_2)$$

$$= \ln\left(\frac{\Gamma+1}{\Gamma-1}\right) + i0$$

$$= \delta + 0i \qquad 0 < \delta \ll 1.$$

$T = \frac{1}{\pi} v$ is a bdded harmonic f^n. satisfying
$T\big|_{v=\pi} = 1$ & $T\big|_{v=0^*} = 0$.

*except at $(0,0)$.

So, $w = \ln\left|\frac{z-1}{z+1}\right| + i \operatorname{Arg}\left(\frac{z-1}{z+1}\right)$ $\quad -\frac{\pi}{2} < \operatorname{Arg}(\cdot) < \frac{3\pi}{2}$.

$$\Rightarrow v = \operatorname{Arg}\left(\frac{z-1}{z+1}\right)$$
$$= \operatorname{Arg}\left(\frac{z-1}{z+1} \cdot \frac{\overline{z+1}}{\overline{z+1}}\right)$$
$$= \operatorname{Arg}\left(\frac{(x-(1-iy))(x+(1-iy))}{(x+1)^2+y^2}\right)$$
$$= \operatorname{Arg}\left(\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}\right)$$

rmk 1 can rewrite in terms of arctan.

So $\boxed{T = \frac{1}{\pi} v \text{ solves the original (D)}}$

rmk 2 isotherms: curves of the form $T(x,y) = C_1$,
$0 < C_1 < 1$ on Ω: circular arcs o.t.f.
$$x^2 + (y - \cot \pi C_1)^2 = \operatorname{cosec}^2(\pi C_1).$$
cf. §120-121 (8Ed §109-§110).

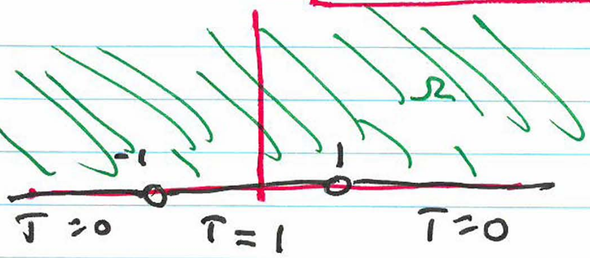
Q^n: what does it mean to say T satisfies the boundary cond^{n}s?

more generally, for the solution of (or, a solution of) (D) $\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$,

what does $u|_{\partial\Omega} = \varphi$ mean?

cf. §135: If φ is continuous & $\partial\Omega$ is "nice"

then: $$\boxed{\lim_{\substack{(x,y)\to(\tilde{x},\tilde{y})\in\partial\Omega \\ (x,y)\in\Omega}} u(x,y) = \varphi(\tilde{x},\tilde{y})}$$



recall $T = \frac{\theta}{\pi} = \frac{1}{\pi}\arg\left(\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}\right)$

$= \frac{1}{\pi}\arctan \frac{2y}{x^2+y^2-1}$

curly a

Here $\arctan\frac{a}{b} = \begin{cases} \arctan a/b & a>0,\ b>0 \\ \pi/2 & a>0,\ b=0 \\ \pi + \arctan a/b & a>0,\ b<0 \end{cases}$

($\arctan(a,b)$ would be better notation).

So, consider e.g.
$$\lim_{y\to0^+} T(-\tfrac{1}{2}, y) = \lim_{y\to0^+}\frac{1}{\pi}\arctan\left(\frac{2y}{y^2-3/4}\right) = 1.$$

Use this to show: $\lim_{\substack{y\to0^+ \\ x\to -1/2}} T(x,y) = 1$.

Similarly $\lim_{\substack{y\to0^+ \\ x\to 2}} T(x,y) = 0$ etc.

Can't expect a limit at $(-1,0)$ or $(1,0)$.