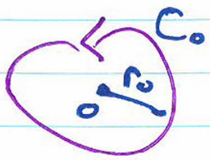
§135 (8 Ed §124) Poisson's integral formula.



Recall Cauchy: if f is analytic in and on C_0, then for $z \in \operatorname{Int} C_0$

$$\boxed{f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds} \qquad (1)$$

Recall for $z = re^{i\theta}$, $r > 0$, the inverse point to z relative to the circle C_0 is

$$r^* e^{i\theta} \quad \text{with} \quad r^* r = r_0^2$$

Recall $(z^*)^* = z$.

Note
$$z^* = r^* e^{i\theta} = \frac{r_0^2}{r} e^{i\theta} = \frac{r_0^2}{re^{-i\theta}} = \frac{r_0^2}{\bar{z}} = \frac{s\bar{s}}{\bar{z}} \quad \text{for any } s \in C_0. \quad (2)$$

Now, fix $z \in \operatorname{Int} C_0$, $z_0 \neq 0$.

Note $\displaystyle\int_{C_0} \frac{f(s)}{s - z^*} ds = 0$ since $s \mapsto \dfrac{f(s)}{s - z^*}$ is analytic in & on C_0, since $z^* \in \operatorname{Ext} C_0$: follows by Cauchy-Goursat.

Combining (1) & (2)

$$f(z) = \frac{1}{2\pi i}\oint_{C_0} \underbrace{\left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*}\right)}_{\mathrm{I}} f(\xi)\, d\xi$$

$$\mathrm{I} = \left(\frac{\xi}{\xi - z} - \frac{\xi}{\xi - z^*}\right)\frac{1}{\xi} \qquad \frac{\xi\bar{\xi}}{\bar{z}}$$

$$= \left(\frac{\xi}{\xi - z} - \frac{1}{1 - \bar{\xi}/\bar{z}}\right)\frac{1}{\xi}.$$

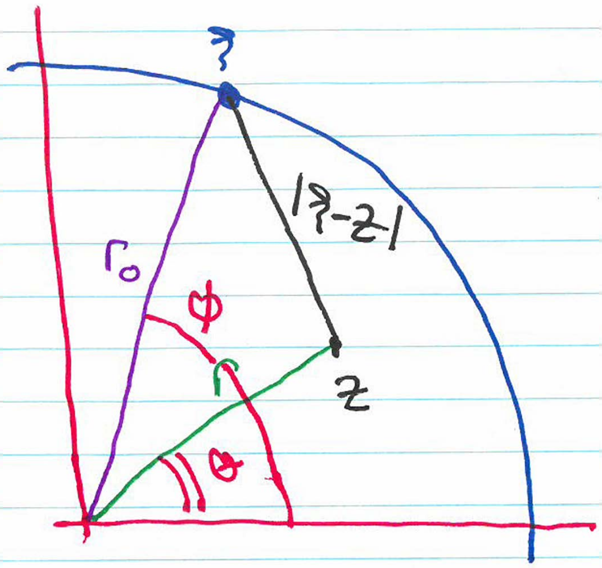
$$= \left(\frac{\xi}{\xi - z} - \frac{\bar{z}}{\bar{z} - \bar{\xi}}\right)\frac{1}{\xi}$$

$$= \frac{\xi\bar{\xi} - z\bar{z}}{|\xi - z|^2}\cdot\frac{1}{\xi} \qquad (4)$$

$z = re^{i\theta}$: put $\xi = r_0 e^{i\phi}$ $\quad 0 \leq \phi \leq 2\pi$

$d\xi = r_0 i e^{i\phi} d\phi$ &

$$(4) \Rightarrow \mathrm{I} = \frac{r_0^2 - r^2}{|\xi - z|^2}\cdot\frac{1}{re^{i\phi}} \qquad (5)$$



cos rule:

$$|\zeta - z|^2 = r_0^2 + r^2 - 2r_0 r\cos(\phi-\theta) \qquad (5)'$$

$$(3) \Rightarrow f(re^{i\theta}) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{r_0^2-r^2}{|\zeta-z|^2} \cdot \frac{1}{\cancel{r_0 e^{i\phi}}} f(r_0 e^{i\phi}) \cancel{r_0 i e^{i\phi}}\, d\phi$$

$$\overset{(5)'}{=} \frac{r_0^2-r^2}{2\pi}\int_0^{2\pi} \frac{f(r_0 e^{i\phi})\, d\phi}{r_0^2 - 2r_0 r\cos(\phi-\theta)+r^2} \qquad (6)$$

Taking the real part of (6) $\Rightarrow$

Given "nice enough" $\Phi(r_0, \phi)$ defined on the bdy C_0 of $B_{r_0}(0)$, a (in fact, the) sol^n^ of the Dirichlet problem

$$(D)\begin{cases} \Delta u = 0 & \text{in } B_{r_0} \\ u|_{\partial B_{r_0}} = \Phi(r_0, \phi) \end{cases}$$

is given by:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2r_0 r\cos(\phi-\theta) + r^2}\,\Phi(r_0,\phi)\,d\phi$$

* Poisson kernel $P(r_0, r, \phi-\theta)$
(Poisson, 1885).

* valid for $r=0$, too ☆.

interpretation of $u|_{\partial B_{r_0}} = \Phi(r_0,\phi)$:

cf. discussion in earlier supplementary material

other Areas of application: electrostatics, fluids.