LECTURE 27

TAYLOR'S THM: in $\mathbb{C}$

Let f be analytic on $B_R(z_0)$.
Then f has a power series representation on $B_R(z_0)$, i.e.,

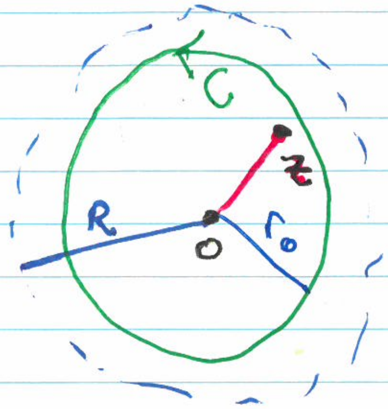
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{for } |z-z_0|<R$$

$$\text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n \in \mathbb{N}_0.$$

(note $f^{(0)} = f$; $0! = 1$).

If $z_0 = 0$, called the Maclaurin series.

<u>pf</u>: Assume $z_0 = 0$, otherwise translate.
Choose $z \in B_R$, $|z| = r$, & fix $r_0 \in (r, R)$, &
set $C = C_{r_0}$, +vely oriented.



Cauchy $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (1)$$

Note
$$\frac{1}{\zeta - z} = \frac{1}{\zeta\underbrace{(1 - z/\zeta)}}$$

note $|z/\zeta| < 1$.

$$= \frac{1}{\zeta}\left(\sum_{n=0}^{N-1} \left(\frac{z}{\zeta}\right)^n + \frac{(z/\zeta)^N}{1 - z/\zeta} \right)$$

$\displaystyle\sum_{n=N}^{\infty} (z/\zeta)^n$ H.G.S. $\displaystyle\frac{(z/\zeta)^N}{1 - z/\zeta}$

$$= \sum_{n=0}^{N-1} \frac{1}{\zeta^{n+1}} z^n + \frac{z^N}{(\zeta - z)\zeta^N}$$

$\times \dfrac{f(\zeta)}{2\pi i}$, integrate over C;

$$(1) \Rightarrow \boxed{f(z) = \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_C \frac{f(\zeta) z^n}{\zeta^{n+1}}\, d\zeta + \frac{z^N}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)\zeta^N}}$$

call this $\rho_{N-1}(z)$ (*)

Extended Cauchy Integral Formula (Lec 22)

$$\Rightarrow f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta$$

So $\circledast \Rightarrow f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0) z^n}{n!} + \rho_{N-1}(z)$

If we can show $\lim_{N\to\infty} \rho_{N-1}(z) = 0$, we're done.

For $\zeta \in C$, there holds $|\zeta| = r_0$.

Suppose (H) $\exists M_N : \left| \frac{f(\zeta)}{(\zeta - z)\zeta^N} \right| \leq M_N$ on C.

Then $|\rho_{N-1}(z)| \leq \frac{r^N}{2\pi} M_N \, l(C)$

$= r_0 r^N M_N$. (**)

To get M_N: f is analytic $\Rightarrow |f|$ is cts.

C is closed & bdd, so extreme value thm. $\Rightarrow$

$\exists \mu$ s.t. $|f| \leq \mu$ on C.

Further, $|\zeta|^N = r_0^N$ &

$|\zeta - z| \geq ||\zeta| - |z|| = r_0 - r$

$\Rightarrow M_N = \frac{\mu}{r_0^N (r_0 - r)}$. This suffices for (H)

Hence, (**) $\Rightarrow |\rho_{N-1}(z)| \leq \frac{\mu r_0 r^N}{r_0^N (r_0 - r)}$

$= \frac{\mu r_0}{r_0 - r} \left(\frac{r}{r_0} \right)^N$

$\to 0$ as $N \to \infty$ since $|r/r_0| < 1$ ☐

In $\mathbb{R}$, it can happen that Taylor series converges, but not to f on any interval about x_0: see lec 15.

Ex 1 $f(z)=e^z$

Th$\stackrel{m}{\cdot}$ $\Rightarrow e^z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$ on $\mathbb{C}$,

noting that $f^{(n)}(z)=e^z\ \forall n \Rightarrow f^{(n)}(0)=e^0=1\ \forall n$.

Verifying $R=\infty$: $\Lambda=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$=\lim_{n\to\infty}\left|\frac{1/(n+1)!}{1/n!}\right|$$

$$=\lim_{n\to\infty}\frac{1}{n+1}=0 \quad \Rightarrow R=\infty.$$

Ex 2 $f(z)=z^2e^{3z}$: find Mac series representation

Ans: f is entire (product of entire f^n_s)

$$\Rightarrow e^{3z}=\sum_{n=0}^{\infty}\frac{3^nz^n}{n!} \quad \text{on } \mathbb{C}$$

$$\Rightarrow z^2e^{3z}=\sum_{n=0}^{\infty}\frac{3^nz^{n+2}}{n!}$$

$n\to n+2$

$$=\sum_{n=2}^{\infty}\frac{3^{n-2}z^n}{(n-2)!}$$

Ex 3 $f(z)=1/_{1-z}$: Find Mac series.

f is analytic for $|z|<1$, indeed for $z\in\mathbb{C}\setminus\{1\}$,

& $f^{(n)}(z)=\frac{n!}{(1-z)^{n+1}} \quad z\neq 1$

$$\Rightarrow f^{(n)}(0)=n!$$

$\Rightarrow$ Taylor series of f at 0, $T_{f,0}$ is given by
$$T_{f,0}(z) = \sum_{n=0}^{\infty} z^n, \quad \text{where this converges.}$$

$$\Lambda = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{1} = 1 \Rightarrow R = 1/1 = 1.$$

Th^m: $\Rightarrow T_{f,0}$ converges to f for $|z|<1$

RMKS ① This follows from geometric series formula.

② Series converges "out to the first singularity", here 1.

Ex 4: Mac series representation for $\frac{1}{2+4z}$

Note f is analytic on $\mathbb{C}\setminus\{-1/2\}$

$$\frac{1}{2+4z} = \frac{1/2}{1+2z} = \frac{1/2}{1-(-2z)} = \frac{1}{2}\,\frac{1}{1-(-2z)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2z)^n \quad \text{for } |-2z|<1$$
i.e., $|z| < 1/2$

$$= \sum_{n=0}^{\infty}(-1)^n 2^{n-1} z^n, \quad \text{for } |z|<1/2.$$

Ex 5 tomorrow: $f(z) = \frac{1+2z^2}{z^3+z^5}$.

what can be done?