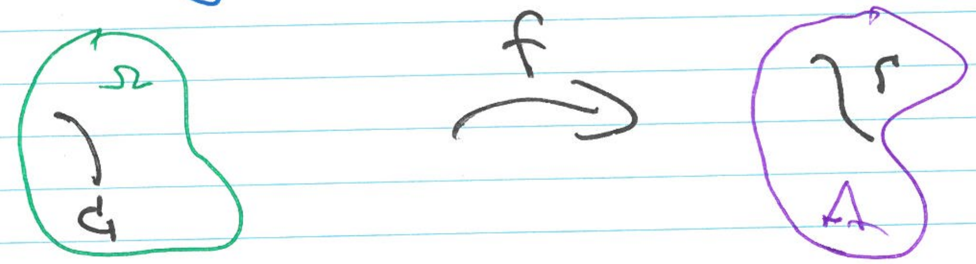
LECTURE 27

Boundary conditions transform similarly.



$\Gamma = f(C)$

$z = x+iy$.

f conformal,

C smooth curve in Ω
i.e., C^∞

Can take C in $\partial\Omega$ with a bit more care.

$H = h(u(x,y), v(x,y))$ is harmonic in Ω if h is harmonic in Λ, &

① Dirichlet b.c. on Γ, i.e., $h(u,v) = \varphi$ on Γ

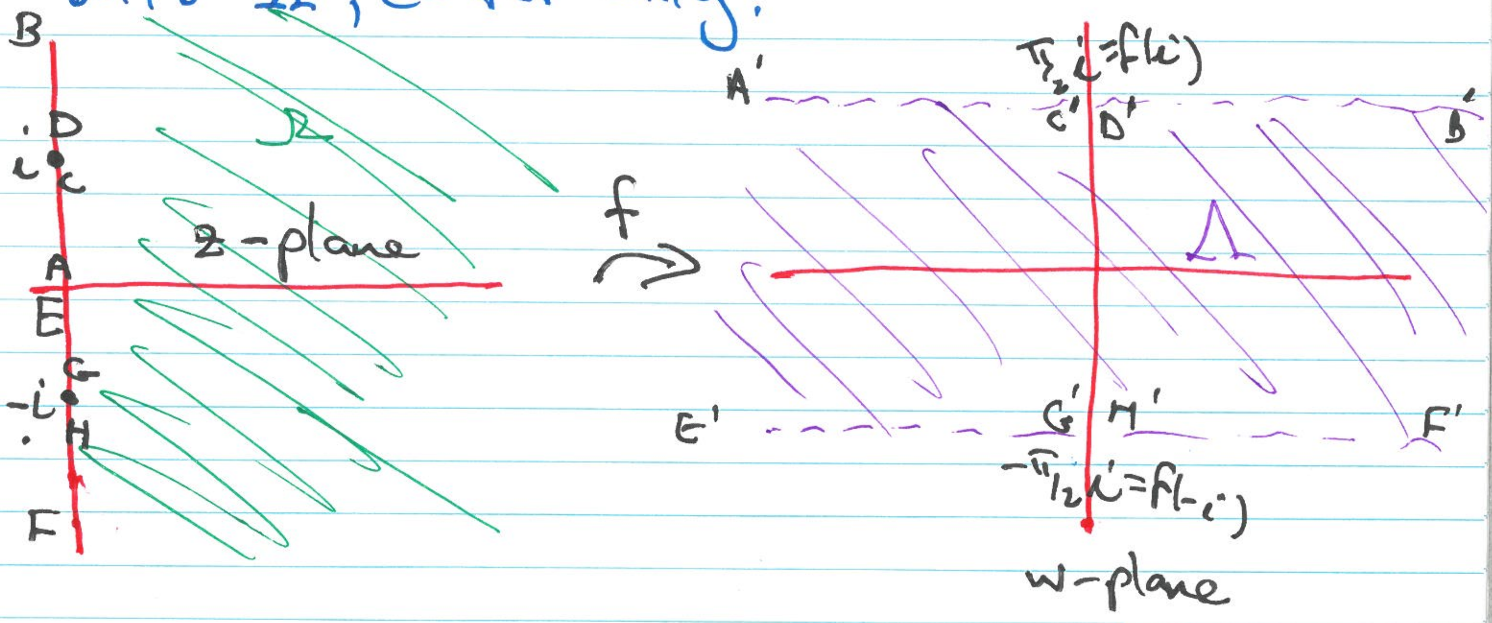
then $\boxed{H(x,y) = \varphi \text{ on } C.}$

② If we have homogeneous Neumann b.c. on Γ, ie $\dfrac{\partial h}{\partial \underline{n}} = 0$ for $\underline{n}$ normal to Γ, then

$$\boxed{\frac{\partial H}{\partial \underline{N}} = 0 \text{ for } \underline{N} \text{ normal to } C.}$$

Example: in $\mathbb{C}$ (w-plane) the f^{n}:
$h(u,v) = v = \operatorname{Im} w$ is harmonic, & in particular, it is harmonic on the 'horizontal' strip Λ, $-\pi/2 < v < \pi/2$.

Claim: $f: z \mapsto \operatorname{Log} z$ maps Ω = Right Half Plane onto Λ, conformally.



$$z \mapsto \operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

A is of the form $0 + \varepsilon i$ $\qquad 0 < \varepsilon << 1$

A' (image of A under f) $= f(A)$

$= \operatorname{Log} A = \ln|A| + i \operatorname{Arg} A$

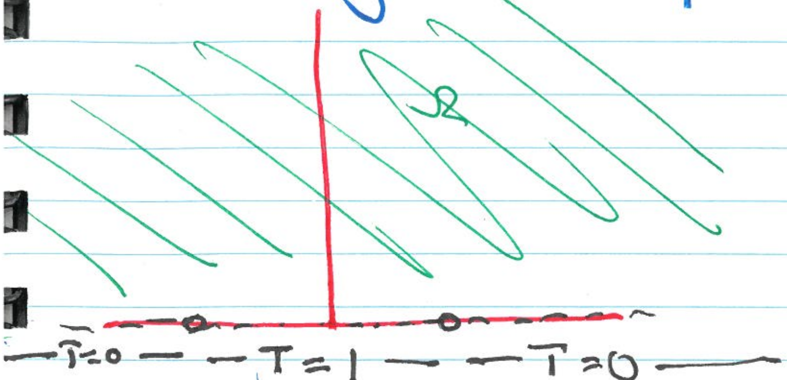
$$= \underbrace{\ln \varepsilon}_{\text{large -ve}} + i \underbrace{\operatorname{Arg}(\varepsilon i)}_{\pi/2}.$$

Conformality: $f'(z) = 1/z$ doesn't vanish on Ω, & f is analytic.

Note $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$
$= \ln\sqrt{x^2+y^2} + i \arctan y/x$ in Ω.

$\Rightarrow H(x,y) = \arctan y/x$ is harmonic in Ω.

Steady-state temperature in a half-plane.



T = temperature.

Want to find the steady state temperature distribution on Ω.

Heat conduction: $\frac{\partial T}{\partial t} = \nabla \cdot (-k \nabla T)$

(t = time) $= -k^2 \Delta T$

Steady state $\Rightarrow$ $\boxed{\Delta T = 0}$

So, want to solve:

$$(D) \begin{cases} \Delta T = 0 \text{ in } \Omega. \\ T(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} (*) \end{cases}$$

(*) chosen to ensure bdy f^n is defined at all points on the bdy. Actual value at ± 1 won't change result.

MATH3201: solve numerically;
MATH 3101: Fourier series;
MATH 3403: Green's function;
MATH 4403: prove sol^n exists (unique).

Physically: look for a bounded sol^n to (D), such that $\lim_{y \to \infty} T(x,y) = 0 \quad \forall x.$