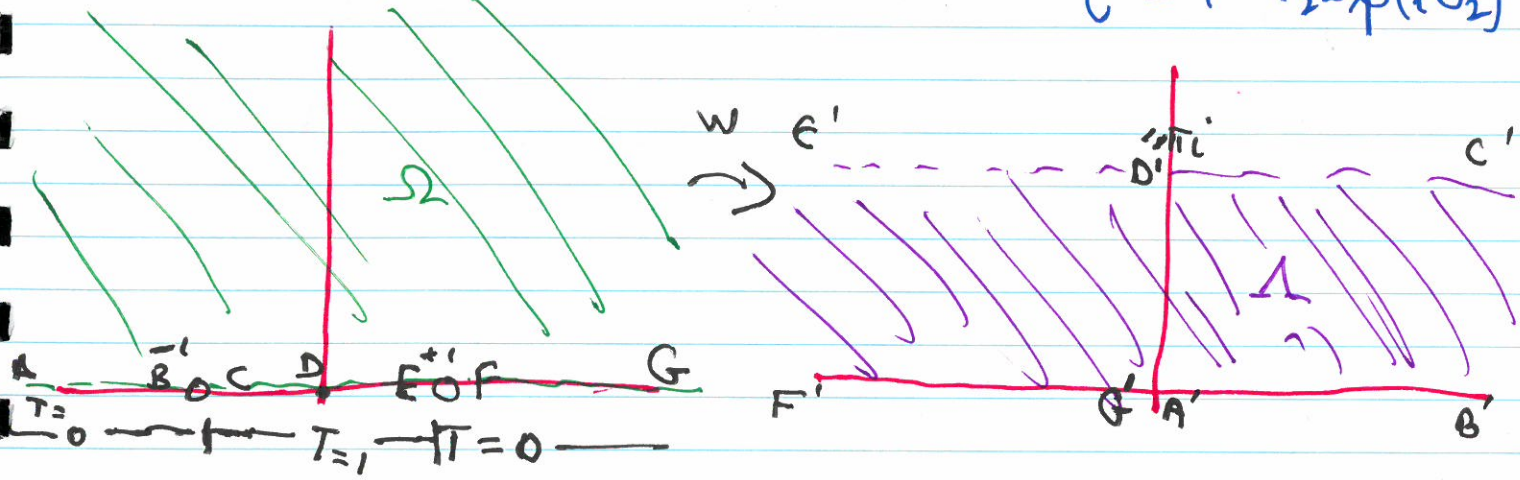
LECTURE 28.

$$w = \operatorname{Log}\left(\frac{z-1}{z+1}\right), \text{ where } -\pi/2 < \arg(\cdot) \leq 3\pi/2.$$

$$= \ln r_1/r_2 + i(\theta_1 - \theta_2) \quad \text{where } \begin{cases} z-1 = r_1 \exp(i\theta_1) \\ z+1 = r_2 \exp(i\theta_2) \end{cases}$$



E.g. $A = -\Gamma + 0i \quad \Gamma \gg 1$

$r_1 = \Gamma + 1, \; r_2 = \Gamma - 1 \qquad \theta_1 = \theta_2 = \pi$

$$\Rightarrow A' = w(A) = \ln \frac{r_1}{r_2} + i(\theta_1 - \theta_2)$$

$$= \ln\left(\frac{\Gamma+1}{\Gamma-1}\right) + i0$$

$$= \delta + 0i \qquad 0 < \delta \ll 1.$$

$T = \frac{1}{\pi} v$ is a bdd harmonic f^n. satisfying
$T|_{v=\pi} = 1$ & $T|_{v=0^*} = 0$.
*except at $(0,0)$.

So, $w = \ln\left|\frac{z-1}{z+1}\right| + i\,\mathrm{Arg}\left(\frac{z-1}{z+1}\right)$ $\quad -\frac{\pi}{2} < \mathrm{Arg}(\cdot) < \frac{3\pi}{2}$.

$$\Rightarrow v = \mathrm{Arg}\left(\frac{z-1}{z+1}\right)$$
$$= \mathrm{Arg}\left(\frac{z-1}{z+1} \cdot \frac{\overline{z+1}}{\overline{z+1}}\right)$$
$$= \mathrm{Arg}\left(\frac{(x-(1-iy))(x+(1-iy))}{(x+1)^2+y^2}\right)$$
$$= \mathrm{Arg}\left(\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}\right)$$

rmk 1 can rewrite in terms of arctan.

So $\boxed{T = \frac{1}{\pi} v \text{ solves the original } D}$

rmk 2 isotherms: curves of the form $T(x,y) = C_1$,
$0 < C_1 < 1$ on Ω: circular arcs o.t.f.
$$x^2 + (y - \cot \pi C_1)^2 = \operatorname{cosec}^2(\pi C_1).$$
cf. §120-121 (8Ed §109–§110).

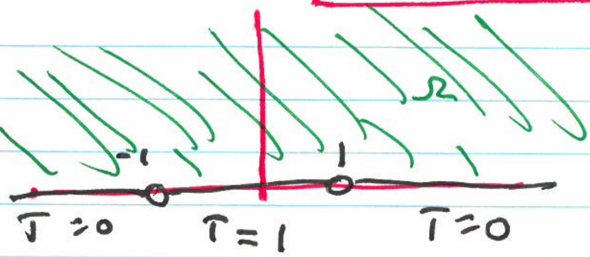
Q^n: what does it mean to say T satisfies the boundary cond^{n}s?

more generally, for the solution of (or, a solution of) (D) $\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$,

what does $u|_{\partial\Omega} = \varphi$ mean?

Cf. §135: If φ is continuous & $\partial\Omega$ is "nice"

then: $\boxed{\lim_{\substack{(x,y)\to(\tilde{x},\tilde{y})\in\partial\Omega \\ (x,y)\in\Omega}} u(x,y) = \varphi(\tilde{x},\tilde{y})}$



recall $T = \frac{v}{\pi} = \frac{1}{\pi}\,\mathrm{arg}\left(\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}\right)$

$= \frac{1}{\pi}\,\mathrm{arctan}\,\frac{2y}{x^2+y^2-1}$

curly a

Here $\mathrm{arctan}\,\frac{a}{b} = \begin{cases} \arctan a/b & a>0,\ b>0 \\ \pi/2 & a>0,\ b=0 \\ \pi + \arctan a/b & a>0,\ b<0 \end{cases}$

($\mathrm{arctan}(a,b)$ would be better notation).

So, consider e.g.

$$\lim_{y\to 0^+} T\left(-\tfrac{1}{2}, y\right) = \lim_{y\to 0^+} \frac{1}{\pi}\,\mathrm{arctan}\left(\frac{2y}{y^2-3/4}\right) = 1.$$

Use this to show: $\lim_{\substack{y\to 0^+ \\ x\to -1/2}} T(x,y) = 1$.

Similarly $\lim_{\substack{y\to 0^+ \\ x\to 2}} T(x,y) = 0$ etc.

Can't expect a limit at $(-1,0)$ or $(1,0)$.

P.S on conformal mappings: scale factor.

$f: z \mapsto w$ conformal (i.e. analytic, $f'(z_0) \neq 0$).

For z near z_0, $f'(z) \neq 0$.

$$\frac{|f(z)-f(z_0)|}{|z-z_0|} \approx |f'(z_0)|$$

$\Rightarrow |f(z)-f(z_0)| \approx |z-z_0|\,|f'(z_0)|$.

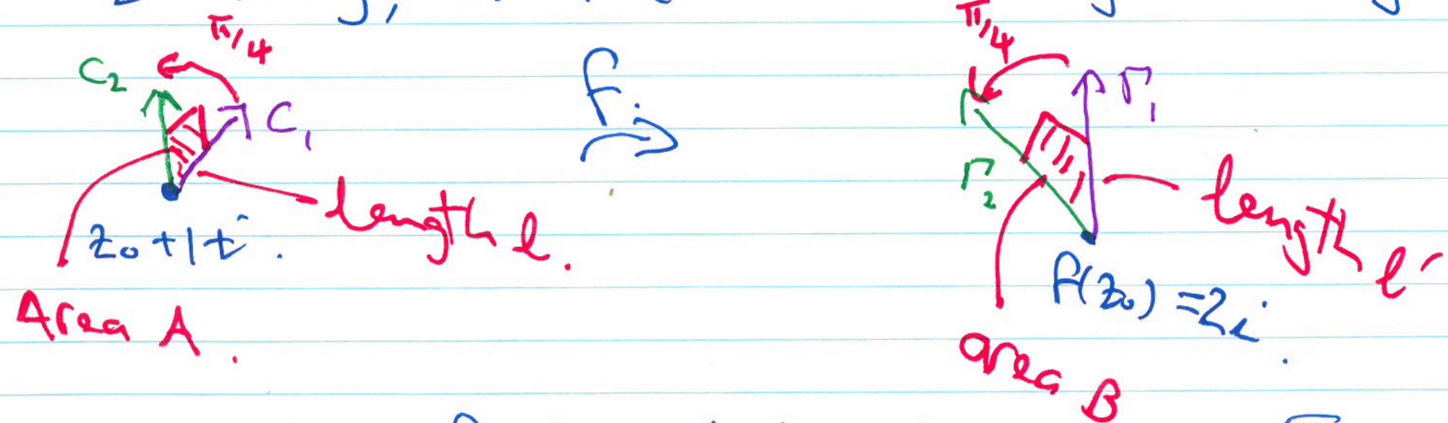
$\Rightarrow |f'(z_0)|$ is the scaling factor or dilation factor.

Stretching is $|f'(z_0)|>1$;

shrinking if $|f'(z_0)|<1$.

E.g. $f(z)=z^2$ at $1+i$.

$z = x+iy$, $w=u+iv$ $\quad u=x^2-y^2 \quad v=2xy$



scaling factor $= |f'(z_0)| = 2|z_0| = 2\sqrt{2}$.

$\Rightarrow l' \approx 2\sqrt{2}\, l$.

& $B \approx (2\sqrt{2})^2 A = 8A$.