LECTURE 28

Ex 5 $\quad f(z) = \dfrac{1+2z^2}{z^3+z^5}$ analytic on $\mathbb{C}\setminus\{0,\pm i\}$.

$$f(z) = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(\frac{2+2z^2}{1+z^2} - \frac{1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

for $|z|<1$, $\dfrac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$

So, for $0<|z|<1$

$$f(z) = \frac{1}{z^3}\left(2-\left(1-z^2+z^4-z^6+\dots\right)\right)$$

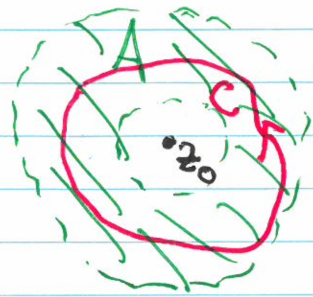
$$= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - \dots$$

Laurent series for f (§66 BBd §60)

Weierstraß 1841 / Laurent 1843.

THM Let f be analytic on the open annulus $A = \{z : r_1 < |z-z_0| < r_2\}$. Let C be a +vely oriented simple closed curve in A, $z_0 \in \text{Int}\, C$.

Then (§67 8th ed §61):



f has a series representation on A:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where $$a_n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^{n+1}}\, d\xi$$

& $$b_n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^{-n+1}}\, d\xi$$

Can rewrite as $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$,

where $$c_n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^{n+1}}\, d\xi$$

In particular, $$c_{-1} = b_1 = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\underbrace{\cancel{(\xi-z_0)^{-1+1}}}_{=1}}\, d\xi$$

$$= \frac{1}{2\pi i}\int_C f(\xi)\, d\xi$$

b_1 is called the residue of f at z_0, denoted $\operatorname*{res}_{z=z_0} f(z)$.

Notes: ① If f is analytic in & on C, then all the b_n's are zero.

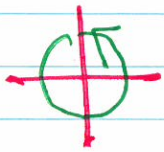
② $r_1 \searrow 0$ & $r_2 \nearrow \infty$ are ok.

③ Taylor & Laurent series are unique (§72, 8Ed §66-67).

Ex6 Find the Laurent series for $e^{1/z} = f(z)$ about zero.

Sol^n: f is analytic on $\mathbb{C}_*$ & (via Taylor series of e^z)

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \cdots \qquad \forall z \neq 0.$$



Note: $\displaystyle \frac{1}{2\pi i}\int_C e^{1/\xi}\, d\xi = b_1 = 1 = \operatorname*{res}_{z=0} f.$

RMKS on series:

① $f(z) = \dfrac{1}{(z-i)^2}$ is a Laurent series about i, with $b_k = \begin{cases} 1 & k=2 \\ 0 & k \neq 2 \end{cases}$ $\quad a_k = 0 \ \forall k.$

② Cauchy product of series (§73, 8Ed §67)

Suppose $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, $g(z) = \sum_{n=0}^{\infty} \beta_n z^n$:

then $(fg)(z) = \sum_{n=0}^{\infty} \gamma_n z^n$, where $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}$

Example: $\dfrac{e^z}{1+z} = e^z \cdot \dfrac{1}{1-(-z)}$

$= (1 + z + z^2/2! + \cdots)(1 - z + z^2 - z^3 + \cdots) \quad |z| < 1$

$= 1 + (-1+1)z + (1 - 1 + \tfrac{1}{2})z^2 + (-1 + 1 - \tfrac{1}{2} + \tfrac{1}{6})z^3 + \cdots$

$= 1 + \dfrac{z^2}{2} - z^3/3 + \cdots$

③ Can take term-by-term integrals & derivatives of such series (§71, 8Ed §65).

§74 (8Ed §68) Singularities

$f: \Omega \to \mathbb{C}$ has a singularity at z_0 if:

① f is not analytic at z_0;

② Given any $\varepsilon > 0$ $\exists z_1 \in B_\varepsilon(z_0)$ s.t. f is analytic at z_1.

f has an isolated singularity at z_0 if f is analytic on $B_\varepsilon(z_0) \setminus \{z_0\}$ for some $\varepsilon > 0$.

Exs
* $1/z$ has an isolated singularity at 0;
* $\operatorname{Log} z$ has non-isolated singularities on $-$ve $\mathbb{R}$e axis $\cup \{0\}$.
* $\dfrac{\sin z}{z}$ has an isolated sing. at 0;
* $\dfrac{1}{\sin(\pi/z)}$ has $\begin{cases} \text{isolated sings at } z = 1/k,\ k \in \mathbb{Z} - \{0\}. \\ \text{non-isolated sing at } 0. \end{cases}$