LECTURE 28

Define $\tilde{\Omega} = \{z : \operatorname{Im} z \geq 0, \ z \neq \pm 1\}$.

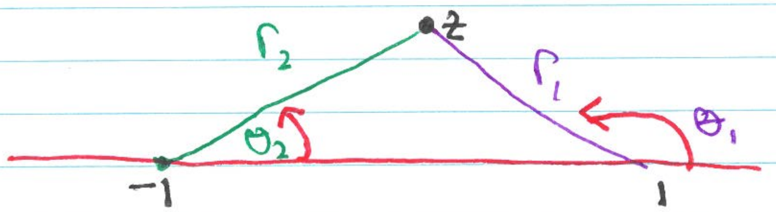
So $\tilde{\Omega} = \Omega \cup \{\text{Re axis}\} \setminus \{-1, 1\}$ $\quad \Omega = \text{UHP}$.

On $\tilde{\Omega}$, define θ_1, θ_2 r_1, r_2 via

$$z - 1 = r_1 \exp(i\theta_1)$$

$$z + 1 = r_2 \exp(i\theta_2)$$

$$r_1, r_2 > 0 \quad , \quad 0 \leq \theta_1, \theta_2 \leq \pi$$



Introduce the transformation

$$w = \operatorname{Log}\left(\frac{z-1}{z+1}\right), \text{ where}$$

Log has a branch cut on the -ve Im axis,

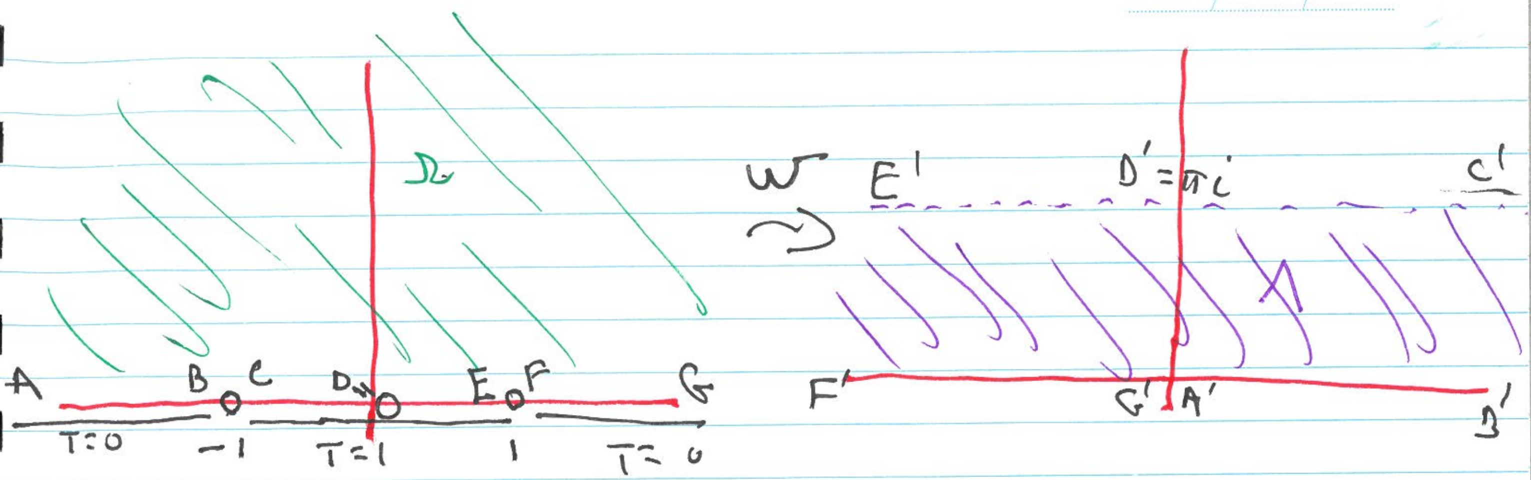
so $-\pi/2 < \operatorname{Arg}(\cdot) \leq 3\pi/2$,

$$w = \operatorname{Log} \frac{r_1 \exp(i\theta_1)}{r_2 \exp(i\theta_2)}$$

$$= \ln \frac{r_1}{r_2} + i(\theta_1 - \theta_2)$$

Claim: w maps Ω onto the horizontal strip $\Lambda = \{u + iv \mid 0 < v < \pi\}$

& $\tilde{\Omega}$ into $\bar{\Lambda}$ as follows:



E.g. $A = -M + 0i$, $M \gg 1$

$r_1(A) = M+1 \qquad r_2(A) = M-1$

$\theta_1 = \theta_2 = \pi$

$$\Rightarrow A' = w(A) = \ln\left(\frac{r_1}{r_2}\right) + i(\theta_1 - \theta_2)$$

$$= \ln\left(\frac{M+1}{M-1}\right) + i0$$

$$= \delta + 0i \qquad \delta \ll 1.$$

$D = 0 + 0i$

$w(D) = \mathrm{Log}\left(\frac{-1}{1}\right) = \ln 1 + i(\pi - 0) = i\pi$

$T = \frac{1}{\pi} v$ is a bded harm f: satisfying

$T|_{v=\pi} = 1$ & $T|_{v=0} = 0$

$$w = \ln\left|\frac{z-1}{z+1}\right| + i\,\mathrm{Arg}\left(\frac{z-1}{z+1}\right) \qquad -\tfrac{\pi}{2} < \mathrm{Arg}(\cdot) \leq \tfrac{3\pi}{2}$$

$$\Rightarrow v = \mathrm{Arg}\left(\frac{z-1}{z+1}\right)$$

$$= \mathrm{Arg}\left(\frac{z-1}{z+1}\cdot\frac{\overline{z+1}}{\overline{z+1}}\right)$$

$$= \mathrm{Arg}\left(\frac{x-(1-iy))(x+(1-iy))}{(x+1)^2+y^2}\right)$$

$$= \mathrm{Arg}\left(\frac{x^2+y^2-1 \;+2iy}{(x+1)^2+y^2}\right).$$

can rewrite in terms of arctan.

Then $T = \frac{1}{\pi}v$ solves the original (D).

<u>Q^n</u>: what does it mean to say T satisfies the bdy cond^{n}s?

More generally, for the solution of (or, a solution of) (D) $\begin{cases}\Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = \varphi\end{cases}$

what does $u|_{\partial\Omega} = \varphi$ mean?

Cf. §135: If φ is cts & $\partial\Omega$ is "nice"

then:
$$\boxed{\lim_{\substack{(x,y)\to(\tilde{x},\tilde{y})\in\partial\Omega \\ (x,y)\in\Omega}} u(x,y) = \varphi(\tilde{x},\tilde{y})}$$

$$T = \frac{V}{\pi} = \frac{1}{\pi} \mathcal{A}rg \frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$$

$$= \frac{1}{\pi} \mathcal{A}rctan \, {}^{2y}/_{x^2+y^2-1} \quad ☆$$

curly a.

Here, $\mathcal{A}rctan \, {}^a/_b = \begin{cases} \arctan {}^a/_b & a>0, b>0 \\ {}^\pi/_2 & a>0, b=0 \\ \pi + \arctan {}^a/_b & a>0, b<0 \end{cases}$

$\arctan(a,b)$ would be better notation.

So consider e.g.

$$\lim_{y \to 0^+} T(-\tfrac{1}{2}, y) = \lim_{y\to 0^+} \frac{1}{\pi} \mathcal{A}rctan\left(\frac{2y}{y^2 - 3/4}\right)$$

$$= 1.$$

Use this to show

$$\lim_{\substack{y \to 0^+ \\ x \to -1/2}} T(x,y) = 1.$$

Can't expect a limit at $(-1,0)$ or $(1,0)$ ☆

P.S. on conformal mappings: scale factor

$f: z \mapsto w$ conformal at z_0 (i.e., analytic, $f'(z_0) \neq 0$)

For z near z_0 $f'(z) \neq 0$

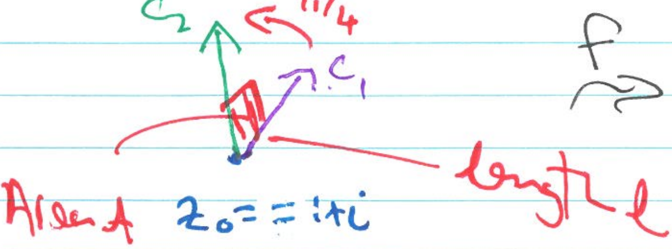
$$\frac{|f(z) - f(z_0)|}{|z - z_0|} \approx |f'(z_0)|$$

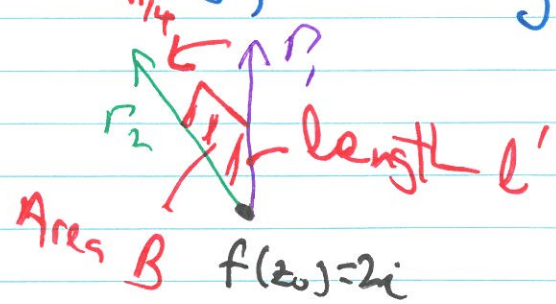
$$\Rightarrow |f(z) - f(z_0)| \approx \underbrace{|f'(z_0)|}_{\text{scaling factor or dilation factor}} \cdot |z - z_0|$$

stretching if $|f'(z_0)| > 1$;
shrinking if $|f'(z_0)| < 1$.

E.g. $f(z) = z^2$ at $1+i$.

$z = x+iy$, $w = u+iv$ $\quad u = x^2 - y^2$, $v = 2xy$





scaling factor $= |f'(z_0)| = 2|z_0| = 2\sqrt{2}$.

$\Rightarrow l' \approx 2\sqrt{2}\, l$

$B \approx (2\sqrt{2})^2 A = 8A$.