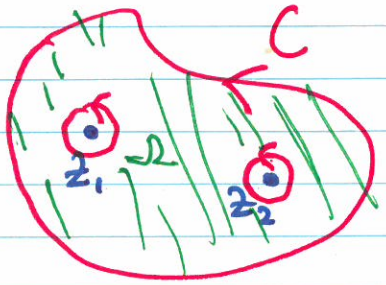
LECTURE 29

(Cauchy) residue theorem

Suppose C is a positively oriented simple closed contour, & that f is analytic on $C \cup \{\text{Int}\, C \setminus \{z_1, \cdots, z_k\}\}$

Then $$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\text{res}}\, f(z)}$$.



Pf Take disjoint, +vely oriented circles $C_1, \ldots, C_k$ around each $z_1, \ldots, z_k$ with disjoint interiors, all lying in $\text{Int}\, C$. Then $C, C_1, \ldots, C_k$ form the bdy of a multiply-connected domain, which we call Ω.

Then, since f is analytic on Ω & $\partial\Omega$, so Cauchy-Goursat extension $\Rightarrow$

$$\underbrace{\int_C f(z)\,dz = \sum_{j=1}^{k} \int_{C_j} f(z)\,dz}_{\text{because } \int_C f + \sum_{j=1}^{k} \int_{-C_j} f = 0} = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\text{res}}\, f(z)$$

cf. Lec 28 defⁿ of b_1.

Classifying isolated singularities

If z_0 is an isolated singularity of f, then $\exists R$ s.t. f has a Laurent series expansion on $B_R(z_0) \setminus \{z_0\}$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

CASE I $b_n = 0 \ \forall n$.

Singularity is removable: setting $f(z_0) = a_0$ makes f analytic on $B_R(z_0)$.

CASE II at least one, but only finitely many of the b_n's are nonzero. Such a singularity is called a pole.

Define $m = \max\{n : b_n \neq 0\}$. m is called the order of the pole.

$$m = 1 \iff \text{simple pole.}$$

CASE III ∞ly many of the b_n's $\neq 0$: called an essential singularity.

Exs of classifying isolated singularities

1. $f(z) = \frac{\sin z}{z}$: analytic on $\mathbb{C}_*$ (quotient of analytic f^{n}s: denom. only vanishes at 0).
$z=0$ is a removable sing. :

$$g(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z=0 \end{cases}$$

is entire, & $g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$.

$$\operatorname*{res}_{z=0} g(z) = 0 = \operatorname*{res}_{z=0} f(z).$$

2. $f(z) = \frac{1}{z^4}$: analytic on $\mathbb{C}_*$, isolated sing at 0, which is a pole of order 4 (largest -ve power in the Laurent series of f, which is just f).

$$\operatorname*{res}_{z=0} f(z) = \text{coeff of } z^{-1} \text{ in Laur. series } = 0$$

3. $h(z) = \frac{\sinh z}{z^4}$: analytic on $\mathbb{C}_*$ (quotient of analytic f^{n}s, denom $=0$ only at 0).
Isolated sing at 0.
On $\mathbb{C}_*$: $h(z) = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$

$$= \frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \cdots$$

So, pole of order 3 at 0: $\operatorname*{res}_{z=0} h(z) = \frac{1}{3!} = \frac{1}{6}$.

RESIDUE at a pole of order m: (§80, 8Ed §73)

<u>THM 1</u>: An isolated sing z_0 is a pole of order $m \geq 1$ iff f can be written near z_0 as:

$$\boxed{f(z) = \frac{\phi(z)}{(z-z_0)^m}}$$ where

* ϕ is analytic on some $B_\varepsilon(z_0)$;

* $\phi(z_0) \neq 0$

<u>THM 2</u>: In this case

$$\boxed{\operatorname*{res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}}$$

$(m-1)$th derivative

In particular, for $m=1$ (simple pole):

$$\boxed{\operatorname*{res}_{z=z_0} f(z) = \frac{\phi^{(0)}(z_0)}{0!} = \phi(z_0)}$$ **

Ex 1 $\quad f(z) = \dfrac{z+i}{z^2+9}$.

f is analytic on $\mathbb{C} \setminus \{\pm 3i\}$ (rational f^n)

$\pm 3i$ are isolated sings.

Near $z = 3i$, write $f(z) = \phi(z)/z-3i$

where $\phi(z) = z+i/z+3i$ is analytic &

non-zero near $z=3i$ ($\phi(3i) = 4i/6i \neq 0$).

Th^m 1 $\Rightarrow$ simple pole at $3i$

Th^m 2 $\Rightarrow$ $\operatorname{res}_{z=3i} f(z) = \phi(3i) = 2/3$.

Near $z = -3i$ ☆.

Ex 2 $\quad g(z) = \dfrac{z^3+2z}{(z-i)^3}$

Analytic on $\mathbb{C} \setminus \{i\}$ (rational f^n).

Near $z=i$, we have $g(z) = \dfrac{\phi(z)}{(z-i)^3}$

where $\phi(z) = z^3+2z$ is entire & $\phi(i) = i \neq 0$.

Th^m 1 $\Rightarrow$ pole of order 3 at i, &

(*) $\Rightarrow$ $\operatorname{res}_{z=i} g(z) = \dfrac{\phi^{(3-1)}(i)}{(3-1)!} = 3i$.