LECTURE 30

To determine the radius of convergence R of $\sum a_n (z-z_0)^n$:

Set $\Lambda = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, then $R = 1/\Lambda$.

$\Lambda = 0 \iff R = \infty$; $\Lambda = \infty \iff R = 0$.

TAYLOR'S THM in $\mathbb{C}$

Let f be analytic on $B_R(z_0)$.
Then f has a power series representation on $B_R(z_0)$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{for } |z-z_0| < R$$

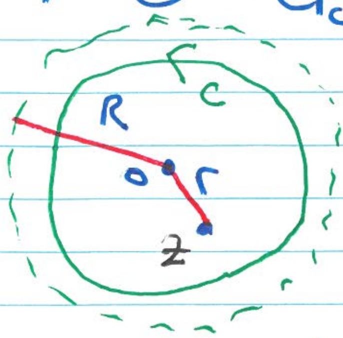
$$\text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n \in \mathbb{N}_0$$

(note $f^{(0)} = f$, $0! = 1$).

If $z_0 = 0$, called the Maclaurin series.

PF: Assume $z_0 = 0$, otherwise translate.

Choose $z \in B_R$, $|z| = r$, fix $r_0 \in (r, R)$ & set $C = C_{r_0}$, +vely oriented.



Cauchy $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta \qquad (1)$$

Note
$$\frac{1}{\zeta - z} = \frac{1}{\zeta(1 - z/\zeta)}$$
note $|z/\zeta| < 1$

$$= \frac{1}{\zeta} \left(\sum_{n=0}^{N-1} (z/\zeta)^n + \frac{(z/\zeta)^N}{1 - z/\zeta} \right)$$

$$= \sum_{n=0}^{N-1} \frac{1}{\zeta^{n+1}} z^n + \frac{z^N}{(\zeta - z)\zeta^N}$$

$\times \dfrac{f(\zeta)}{2\pi i}$, integrate over C.

call this $\rho_{N-1}(z)$

$$(1) \Rightarrow f(z) = \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_C \frac{f(\zeta) z^n}{\zeta^{n+1}} d\zeta + \frac{z^N}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)\zeta^N} d\zeta \qquad (*)$$

Extended Cauchy integral formula (Lec 23) $\Rightarrow f^{(n)}(0) = \dfrac{n!}{2\pi i} \displaystyle\int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$

So, $(*) \Rightarrow f(z) = \displaystyle\sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_{N-1}(z)$

If we can now show $\lim_{N \to \infty} \rho_{N-1}(z) = 0$,

we're done.

For $\zeta \in C$, note $|\zeta| = r_0$.

Suppose (H) $\exists M_N : \left|\frac{f(\zeta)}{(\zeta - z)\zeta^N}\right| \leq M_N$ on C.

Then $|\rho_{N-1}(z)| \leq \frac{r^N}{2\pi} M_N \, l(C_1)$

$= r_0 r^N M_N$. (**)

To get M_N: f is analytic $\Rightarrow |f|$ is cts, C is closed & bdd, so extreme value th^m $\Rightarrow \exists \mu$ s.t. $|f| \leq \mu$ on C.

Further, $|\zeta|^N = r_0^N$ &

$|\zeta - z| \geq ||\zeta| - |z|| = r_0 - r$.

$\Rightarrow M_N = \frac{\mu}{r_0^N (r_0 - r)}$. This suffices for (H)

Hence, (**) $\Rightarrow |\rho_{N-1}(z)| \leq \frac{\mu r_0 r^N}{r_0^N (r_0 - r)}$

$= \frac{\mu r_0}{r_0 - r} \left(\frac{r}{r_0}\right)^N$

$\to 0$ as $N \to \infty$ since $\left|\frac{r}{r_0}\right| < 1$. $\square$

In $\mathbb{R}$ Taylor series may converge, but fail to converge to the f^n on any interval about z_0, cf. Lec 16

Ex 1 $\quad f(z) = e^z$.

$\text{Th}^{m}! \Rightarrow e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ on $\mathbb{C}$,

noting that $f^{(n)}(z) = e^z \ \forall n$

$\Rightarrow f^{(n)}(0) = e^0 = 1 \ \forall n$.

Verifying $R = \infty$: $\Lambda = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \left|\frac{1/(n+1)!}{1/n!}\right|$$

$$= \lim_{n\to} 1/n+1 = 0 \Rightarrow R = \infty.$$

Ex 2 $\quad f(z) = z^2 e^{3z}$: find Mac series.

Ans: f is entire (product of entire f^{n}s).

$$\Rightarrow e^{3z} = \sum_{n=0}^{\infty} \frac{3^n z^n}{n!} \quad \text{on } \mathbb{C}.$$

$$\Rightarrow z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n z^{n+2}}{n!}$$

$n \to n+2$

$$= \sum_{n=2}^{\infty} \frac{3^{n-2} z^n}{(n-2)!}$$

Ex 3 : on Friday.