LECTURE 31

TAYLORS THM in $\mathbb{C}$

Let f be analytic on $B_R(z_0)$.
Then f has a power series representation on $B_R(z_0)$.

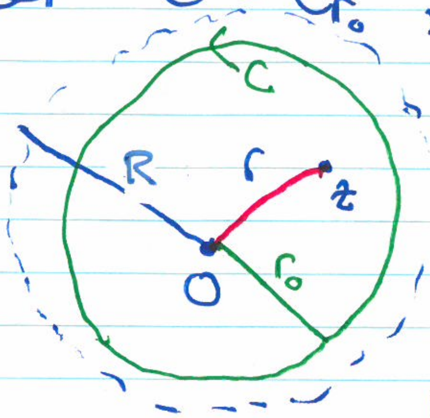
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{for } |z-z_0| < R$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ $\quad n \in \mathbb{N}_0$

(note $f^{(0)} = f$; $0! = 1$)

If $z_0 = 0$, called Maclaurin series.

Pf: Assume $z_0 = 0$, otherwise translate.
Choose $z \in B_R$, $|z| = r$, & fix $r_0 \in (r, R)$ & set $C = C_{r_0}$, +vely oriented.



Cauchy $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} \, d\xi \quad (1)$$

Note: $$\frac{1}{\xi - z} = \frac{1}{\xi \underbrace{(1 - z/\xi)}_{\text{note: } |z/\xi| < 1}}$$

$$= \frac{1}{\xi} \left(\sum_{n=0}^{N-1} (z/\xi)^n + \frac{(z/\xi)^N}{1 - z/\xi} \right).$$

$$= \sum_{n=0}^{N-1} \frac{1}{\xi^{n+1}} z^n + \frac{z^N}{(\xi - z)\xi^N}$$

$\times$ by $\frac{f(\xi)}{2\pi i}$, integrate over C;

$$(1) \Rightarrow f(z) = \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_C \frac{f(\xi) z^n}{\xi^{n+1}} \, d\xi + \frac{z^N}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)\xi^N} \, d\xi \quad (*)$$

call this $\rho_{N-1}(z)$

extended Cauchy integral formula (Lec 23)

$$\Rightarrow f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{\xi^{n+1}} \, d\xi$$

So, $\textcircled{*} \Rightarrow f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0) z^n}{n!} + \rho_{N-1}(z)$

If we can show $\lim_{N \to \infty} \rho_{N-1}(z) = 0$, we're done.

For $\xi \in C$, there holds $|\xi| = r_0$.

Suppose (H) $\exists M_N : \left| \frac{f(\xi)}{(\xi - z)\xi^N} \right| \leq M_N$ on C.

Then $|\rho_{N-1}(z)| \leq \frac{r^N}{2\pi} M_N \, \ell(C)$

$= r_0 r^N M_N$. (**)

To get M_N: f is analytic $\Rightarrow |f|$ is cts, C is closed & bded, so extreme value th^m $\Rightarrow \exists \mu$ s.t. $|f| \leq \mu$ on C.

Further, $|\xi|^N = r_0^N$ &

$|\xi - z| \geq \big| |\xi| - |z| \big| = r_0 - r$.

$\Rightarrow M_N = \frac{\mu}{r_0^N (r_0 - r)}$. This suffices for (H).

Hence, (**) $\Rightarrow |\rho_{N-1}(z)| \leq \frac{\mu r_0 r^N}{r_0^N (r_0 - r)}$

$= \frac{\mu r_0}{r_0 - r} \left(\frac{r}{r_0} \right)^N$

$\to 0$ as $N \to \infty$ since $|r/r_0| < 1$ $\square$

In $\mathbb{R}$, Taylor series may converge, but fail to converge to the f^n: cf Lec. 16.

Ex 1 $f(z) = e^z$

Th^{m}: $\Rightarrow e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ on $\mathbb{C}$ ☺

noting that $f^{(n)}(z) = e^z \;\forall n \Rightarrow f^{(n)}(0) = e^0 = 1 \;\forall n$.

Verifying $R = \infty$: $\Lambda = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{1/(n+1)!}{1/n!} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 \quad \Rightarrow R = \infty.$$

Ex 2 $f(z) = z^2 e^{3z}$: find Mac series.

Ans: f is entire (product of entire f^{n}s):

☺ $\Rightarrow e^{3z} = \sum_{n=0}^{\infty} \frac{3^n z^n}{n!}$ on $\mathbb{C}$

$$\Rightarrow z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n z^{n+2}}{n!}$$

$n \to n+2$

$$= \sum_{n=2}^{\infty} \frac{3^{n-2} z^n}{(n-2)!}$$

Ex 3 $f(z) = 1/_{1-z}$: find Mac series.

f is analytic for $|z| < 1$, indeed for $z \in \mathbb{C} \setminus \{1\}$,

& $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ $\quad z \neq 1$

$\Rightarrow f^{(n)}(0) = n!$

$\Rightarrow$ Taylor series of f at 0, $T_{f,0}$ is given by

$T_{f,0}(z) = \sum_{n=0}^{\infty}$ where this converges.

$$\Lambda = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{1} = 1 \Rightarrow R = \frac{1}{1} = 1.$$

Th^m: $\Rightarrow T_{f,0}$ converges to f for $|z|<1$.

RMKS: 1 This follows for geom series formula.

2 Series converges "out to the first singularity", here 1.

Ex 4: Mac series for $\frac{1}{2+4z}$

Note f is analytic on $\mathbb{C}\setminus\{-\frac{1}{2}\}$

$$\frac{1}{2+4z} = \frac{1/2}{1+2z} = \frac{1/2}{1-(-2z)} = \frac{1}{2}\,\frac{1}{1-(-2z)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2z)^n \quad \text{for } |-2z|<1$$

i.e., $|z|<\frac{1}{2}$.

$$= \sum_{n=0}^{\infty}(-1)^n 2^{n-1} z^n \quad \text{for } |z|<\tfrac{1}{2}.$$

Ex 5 Tomorrow $f(z) = \frac{1+2z^2}{z^3+z^5}$:

what can be done?