LECTURE 31

Example: consider $f(z) = \cot z = \dfrac{\cos z}{\sin z}$.

$f = \dfrac{p}{q}$ $p(z) = \cos z$, $q(z) = \sin z$ are entire, so sings of f are zeros of sin, i.e., $n\pi$, $n \in \mathbb{Z}$.

Note $p(n\pi) = (-1)^n$ $n \in \mathbb{Z}$

$q(n\pi) = 0$

$q'(n\pi) = \cos(n\pi) = (-1)^n \neq 0$.

So Lec 30, Th: 2 $\Rightarrow$ each sing $z_n = n\pi$ of f is a simple pole, with:

$$\operatorname*{res}_{z=z_n} f(z) = \frac{p(z_n)}{q'(z_n)} = \frac{(-1)^n}{(-1)^n} = 1.$$

Revisit exs from Lec 29 with these techniques.

PS on zeros of analytic f^{n}s:

Th^m (cf. Th^m 2 §82) (8th ed §75).

Zeros of analytic f^{n}s are isolated unless $f \equiv 0$.

Pf: Suppose f is analytic at z_0, $f(z_0)=0$.

Lec 28 $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Let $m = \min n : a_n \neq 0$. $m \geq 1$ since $f(z_0) \neq 0$, & $m < \infty$ since $f \not\equiv 0$.

$\Rightarrow f$ has a zero of order m at z_0.

Th^m 1 §82 $\Rightarrow f(z) = (z-z_0)^m g(z)$ on some nbhd of z_0, where g is analytic & $g(z_0) \neq 0$. Since g is analytic it is cts, so on some (possibly smaller) nbhd, $g \neq 0$

$\Rightarrow f(z) \neq 0$ for $0 < |z-z_0| < \varepsilon$ for some $\varepsilon > 0$.

□

Key application of Res Th^{m}:
using contour integrals in $\mathbb{C}$ to evaluate integrals over $\mathbb{R}$ (typically from $-\infty$ to ∞ or 0 to ∞).

Recall $\circledast$ $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{M_1\to\infty}\int_{M_1}^{0} f(x)\,dx + \lim_{M_2\to\infty}\int_{0}^{M_2} f(x)\,dx,$$
if both the limits on the RHS exist.

Rmk: you can replace 0 by any fixed $c\in\mathbb{R}$.

You cannot in general replace RHS of $\circledast$ by
$$\lim_{M\to\infty}\int_{-M}^{M} f(x)\,dx.$$

If you do this anyway, it defines Cauchy Principal Value (PV) integral.

E.g. $$\int_{-\infty}^{\infty} x\,dx = \lim_{M_1\to-\infty}\int_{M_1}^{0} x\,dx + \lim_{M_2\to\infty}\int_{0}^{M_2} x\,dx$$
$$= \lim_{M_1\to\infty} -M_1^2/2 + \lim_{M_2\to\infty} M_2^2/2$$
$$= \text{undefined, but}$$

$$PV\int_{-\infty}^{\infty} x\,dx = \lim_{M\to\infty}\int_{-M}^{M} x\,dx$$
$$= \lim_{M\to\infty}\left[M^2/2 - (-M)^2/2\right]$$
$$= \lim_{M\to\infty}[0]$$
$$= 0.$$

When is $PV\int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} f$?

E.g for f even, or for $f \geq 0$.

In particular, for f even, i.e., $f(x)=f(-x)$ $\forall x \in \mathbb{R}$, we have

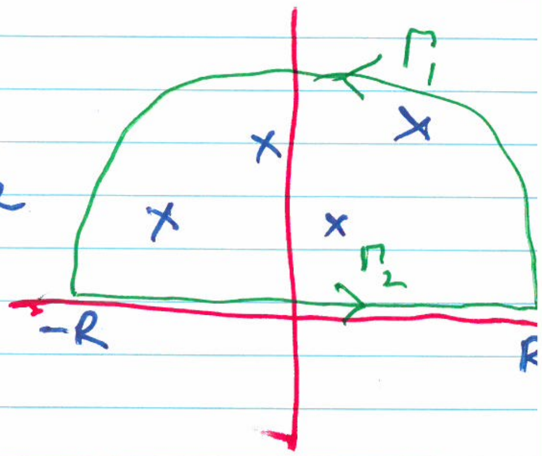
$$\int_0^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{2} PV\int_{-\infty}^{\infty} f(x)\,dx, \ \&$$

all of these converge or diverge together.

How does this fit with contour integrals?

Suppose f is even & "nice" on $\mathbb{R}$

Aim: evaluate $\int_{-\infty}^{\infty} f$



Suppose f is analytic inside & on the contour $C=\Gamma_1+\Gamma_2$ except maybe for isolated singularities in Int C.

$$\int_C f = \underbrace{\int_{\Gamma_1} f}_{\mathrm{I}} + \underbrace{\int_{\Gamma_2} f}_{\mathrm{II}}$$

LHS: hopefully can evaluate via Res Th^m.

Let $R \to \infty$: $\mathrm{II} \xrightarrow[R\to\infty]{} PV\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$,

since f is even.

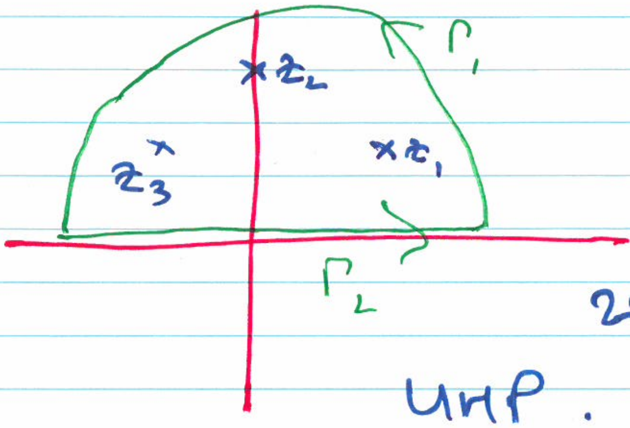
Only remains to deal with $\lim_{R\to\infty}\int_{\Gamma_1} f$:

hopefully $=0$, e.g., via M-L. estimate.

Example: evaluate $I = \int_0^\infty \overbrace{\frac{x^2}{1+x^6}}^{f(x)}\,dx$

Note f is even, & cts on $\mathbb{R}$, & $f \sim 1/x^4$ as $x \to \pm\infty$, so I converges (p-test, $p>1$).

Note $f(z) = \dfrac{z^2}{1+z^6}$ is a rational f^n, & is analytic on $\mathbb{C}$ except for the 6 zeros of the denominator, i.e., sol^{n}s of $z^6+1=0$.



$R>1$: f is analytic in & on $C = \Gamma_1+\Gamma_2$, except for the 3 zeros of z^6+1 in the UHP.

$$e^{\pi i/6} \qquad i \qquad e^{5\pi i/6}$$

$$z_1 \qquad z_2. \qquad z_3.$$

Res th^m $\Rightarrow$ $$\int_C f = 2\pi i \sum_{j=1}^{3} \operatorname*{res}_{z=z_j} f(z) \qquad (1)$$

Note f has the form p/q: for each z_j, $j=1,2,3$ there holds

$$p(z_j) = z_j^2 \neq 0$$

$$q(z_j) = 1+z_j^6 = 0$$

$$q'(z_j) = 6z_j^5 \neq 0$$

$\Rightarrow$ simple pole at each z_j..