LECTURE 31

Ex 3 $f(z) = \dfrac{1}{1-z}$: find Mac series.

f is analytic for $|z|<1$, indeed for $z \in \mathbb{C} \setminus \{1\}$.

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \qquad z \neq 1.$$

$$\Rightarrow f^{(n)}(0) = n!$$

Taylor series of f at 0, $T_{f,0}$ is given by

$$T_{f,0}(z) = \sum_{n=0}^{\infty} z^n, \quad \text{where this converges.}$$

$$\Lambda = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{1} = 1, \text{ so } R = 1/\Lambda = 1/1 = 1.$$

Thm Lec 30 $\Rightarrow$ $T_{f,0}$ converges to f for $|z|<1$.

RMK : 1 This follows from geometric series formula.

2 Series converges "out to the first singularity", which here is 1.

Ex 4 $\dfrac{1}{2+4z}$: find Mac series.

Note f is analytic on $\mathbb{C} \setminus \{-\tfrac{1}{2}\}$.

$$\frac{1}{2+4z} = \frac{1/2}{1+2z} = \frac{1/2}{1-(-2z)} = \frac{1}{2}\,\frac{1}{1-(-2z)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (-2z)^n \qquad \text{for } |-2z|<1$$

i.e., $|z|<\tfrac{1}{2}$.

$$= \sum_{n=0}^{\infty} (-1)^n 2^{n-1} z^n \qquad \text{for } |z|<\tfrac{1}{2}.$$

Ex 5 $\quad f(z) = \dfrac{1+2z^2}{z^3+z^5}$ analytic on $\mathbb{C} \setminus \{0, \pm i\}$

$$f(z) = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(\frac{2+2z^2}{1+z^2} - \frac{1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

for $|z|<1$ $\quad \dfrac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

So for $0<|z|<1$

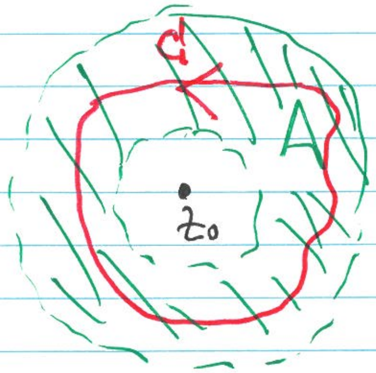
$$f(z) = \frac{1}{z^3}\left(2 - (1 - z^2 + z^4 - z^6 + \dots)\right)$$

$$= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - \dots .$$

Laurent series for f (§66, 8Ed §60)

Weierstraß 1841, Laurent 1843.

<u>THM</u>: Let f be analytic on the open annulus $A = \{ z : r_1 < |z - z_0| < r_2 \}$. Let C be a simple closed curve in A, $z_0 \in \operatorname{Int} C$. Note $\operatorname{Int} C \subset \mathbb{C}$.



Then (§67, 8Ed §61): f has a series representation on A:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$

$b_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{-n+1}} d\xi$.

Can rewrite as $f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$

where $c_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$

In particular, $c_{-1} = b_1 = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\underbrace{\cancel{(\xi - z_0)^{-1+1}}}_{1}} d\xi$

$= \frac{1}{2\pi i} \int_C f(\xi) d\xi$

This is called the residue of the function at z_0, denoted $\operatorname*{res}_{z = z_0} f(z)$.

Notes: ① If f is analytic in & on C, then all the b_n's are zero.

② $r_1 \downarrow 0$ &/or $r_2 \uparrow \infty$ are ok.

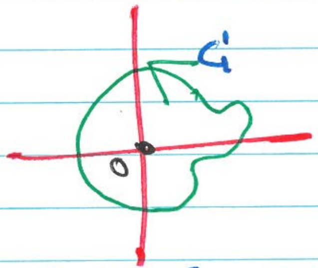
③ Taylor & Laurent series are unique (§72, 8Ed §66–67).

Ex 6 Find the Laurent series of $f(z)=e^{1/z}$ about zero.

f is analytic on $\mathbb{C}_*$ & (via Taylor series of e^z, Ex 1):

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} = 1 + \overset{\text{"}b_1}{\frac{1}{z}} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$$

$\forall z \neq 0$. Note $\frac{1}{2\pi i}\int_{C_1} e^{1/\zeta}\, d\zeta = b_1 = 1$.



RMKS on Series:

① $f(z) = \frac{1}{(z-i)^2}$ is a Laurent series about i, with $b_k = \begin{cases} 1 & k=2 \\ 0 & k\neq 2 \end{cases}$, $a_k = 0\ \forall k$.

② Cauchy product of series (§73, 8Ed §67)

Suppose $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, $g(z) = \sum_{n=0}^{\infty} \beta_n z^n$.

Then $(fg)(z) = \sum_{n=0}^{\infty} \gamma_n z^n$, where $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}$

Example: $\dfrac{e^z}{1+z} = \dfrac{e^z}{1-(-z)}$

$= \left(1 + z + z^2/2! + \cdots\right)\left(1 - z + z^2 - z^3 + \cdots\right) \quad |z|<1$

$= 1 + (-1+1)z + (1-1+1/2)z^2 + (-1+1-1/2+1/6)z^3 + \cdots$

$= 1 + \dfrac{z^2}{2} - \dfrac{z^3}{3} + \cdots$

(3) Can take term-by-term derivatives & integrals of such series (§71, 8Ed §65).

Next: Singularities.