# LECTURE 32

**Ex 5** $f(z) = \dfrac{1+2z^2}{z^3+z^5}$ analytic on $\mathbb{C} \setminus \{0, \pm i\}$.

$$f(z) = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(\frac{2+2z^2}{1+z^2} - \frac{1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

for $|z|<1$, $\dfrac{1}{1+z^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n z^{2n}$

So, for $0<|z|<1$

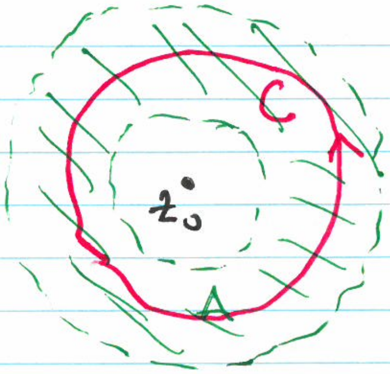
$$f(z) = \frac{1}{z^3}\left(2 - (1 - z^2 + z^4 - z^6 + \dots)\right)$$

$$= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - \dots$$

Laurent series for $f$ (§66, 8Ed §60)

Weierstraß 1841 / Laurent 1843.

THM: Let $f$ be analytic on the open annulus $A = \{z : r_1 < |z - z_0| < r_2\}$. Let $C$ be +vely oriented simple closed curve in $A$, $z_0 \in \operatorname{Int} C$. Note $\operatorname{Int} C \subset \mathbb{C}$.




Then (§67, 8Ed §61):

$f$ has a series representation on $A$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{on } A,$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$

& $b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}}\, d\zeta$

Can rewrite as $f(z) = \displaystyle\sum_{-\infty}^{\infty} c_n (z - z_0)^n$,

where $c_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$.

In particular, $c_{-1} = b_1 = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(\zeta)}{\underbrace{\cancel{(\zeta - z_0)^{-1+1}}}_{=1}}\, d\zeta$

$$= \frac{1}{2\pi i} \int_C f(\zeta)\, d\zeta$$

This is called the residue of $f$ at $z_0$, denoted $\operatorname*{res}_{z = z_0} f(z)$.

Note: ① If $f$ is analytic in & on $C$, then all the $b_n$'s are zero.

② $r_1 \downarrow 0$ & $r_2 \uparrow \infty$ are ok.

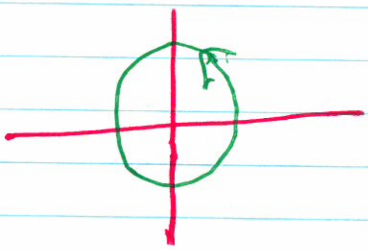
③ Taylor & Laurent series are unique (§72, 8Ed §66-67).

**Ex6** Find the Laurent series for $f(z) = e^{1/z}$ about zero.

$f$ is analytic on $\mathbb{C}_*$ & (via Taylor series of $e^z$)

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \frac{1}{3!\, z^3} + \cdots$$

$\forall\, z \neq 0$.



Note: $\frac{1}{2\pi i} \int_C e^{1/\zeta} d\zeta = b_1 = 1$.

**Ex7** Compute $I = \int_C \frac{5z-2}{z(z-1)} dz$ $\quad C = 2e^{i\theta}$, $\theta \in [0, 2\pi]$.

Note: $f$ is analytic on $\mathbb{C} \setminus \{0,1\}$. Soon.

RMKS on series:

① $f(z) = \frac{1}{(z-i)^2}$ is a Laurent series about $i$, with $b_k = \begin{cases} 1 & k=2 \\ 0 & k \neq 2 \end{cases}$; $a_k = 0 \;\forall k$.

② Cauchy product of series (§73, 8Ed §67)

Suppose $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, $g(z) = \sum_{n=0}^{\infty} \beta_n z^n$:

then $(fg)(z) = \sum_{n=0}^{\infty} \gamma_n z^n$, where $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}$

Example: $\dfrac{e^z}{1+z} = e^z \dfrac{1}{1-(-z)}$

$= \left(1 + z + z^2/2! + \cdots\right)\left(1 - z + z^2 - z^3 + \cdots\right)$ $|z|<$

$= 1 + (-1+1)z + (1-1+\tfrac{1}{2})z^2 + (-1+1-\tfrac{1}{2}+\tfrac{1}{6})z^3 + \cdots$

$= 1 + \dfrac{z^2}{2} - \dfrac{z^3}{3} + \cdots$

(3) can take term-by-term derivatives & integrals of such series (§71, 8Ed §65).

## §74 (8Ed §68) Singularities

$f: \Omega \to \mathbb{C}$ has a singularity at $z_0$ if:

(1) $f$ is not analytic at $z_0$;

(2) Given any $\varepsilon > 0$, $\exists z_1 \in B_\varepsilon(z_0)$ s.t. $f$ is analytic at $z_1$.

$f$ has an isolated singularity at $z_0$ if $f$ is analytic on $B_\varepsilon(z_0) \setminus \{z_0\}$ for some $\varepsilon > 0$