LECTURE 32

Ex 1, ctd:

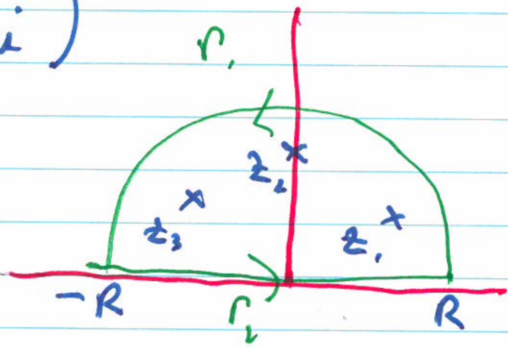
recall: Res th^{m} $\Rightarrow$ $\displaystyle\int_C f = 2\pi i \sum_{j=1}^{3} \operatorname{res}_{z=z_j} f(z)$ (1)'

RHS of (1)' $\displaystyle = 2\pi i \sum_{j=1}^{3} \frac{p(z_j)}{q'(z_j)}$

$$= 2\pi i \sum_{j=1}^{3} z_j^2 / 6z_j^5 = 2\pi i \sum_{j=1}^{3} 1/6z_j^3$$

$$= 2\pi i \left(1/6i - 1/6i + 1/6i \right)$$

$$= \pi/3$$



Note: $\displaystyle\int_C = \underbrace{\int_{\Gamma_1} f}_{(\alpha)} + \underbrace{\int_{\Gamma_2} f}_{(\beta)}$ (2)'

As $R \to \infty$, $(\beta) \to 2I$.

Claim $(\alpha) \to 0$ as $R \to \infty$.

Note: $|(\alpha)| \leq M_R \ell_R$ (*)'

where $\ell_R = \text{length}(\Gamma_1) = \pi R$ &

$$M_R = \max_{z \in \Gamma_1} |f(z)|$$

$$\leq \max_{|z|=R} \left| \frac{z^2}{z^6+1} \right|$$

$$\overset{R>1}{\leq} \frac{R^2}{R^6-1}$$ via reverse Δ ineq, $|a+b| \geq \big||a|-|b|\big|$

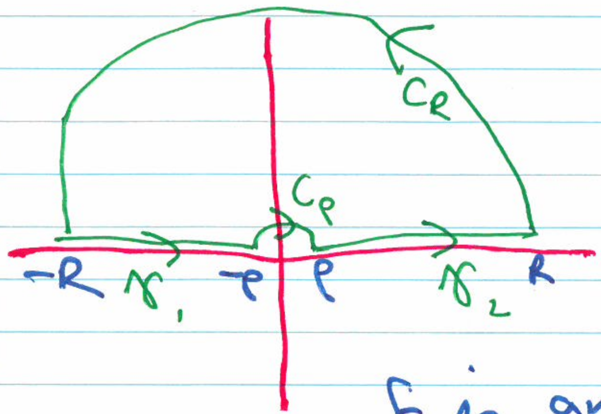
So (*)' $\Rightarrow$ $\displaystyle|(\alpha)| \leq \frac{\pi R \cdot R^2}{R^6-1} = \frac{\pi}{R^3 - 1/R^3} \to 0$ as $R \to \infty$.

So $\lim_{R \to \infty}$ in (2)' $\Rightarrow$ $\pi/3 = 0 + 2I \Rightarrow I = \pi/6$. $\square$

Ex 2 $\quad I = \int_0^\infty \frac{\sin x}{x}\,dx.$

Note that $\frac{\sin x}{x}$ is even, so if I exists,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\,PV\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx.$$



Take $f(z) = e^{iz}/z$

$C = \gamma_1 + C_\rho + \gamma_2 + C_R.$

f is analytic on $\mathbb{C}_*$ & in particular on C & Int C, so $\int_C f = 0$ by Cauchy-Goursat $\Rightarrow$

$$\underbrace{\int_{\gamma_1} f}_{I_1} + \underbrace{\int_{C_\rho} f}_{I_2} + \underbrace{\int_{\gamma_2} f}_{I_3} + \underbrace{\int_{C_R} f}_{I_4} = 0 \qquad (+)$$

$$I_1 = \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx \overset{w=-x}{=} -\int_{\rho}^{R} \frac{e^{-iw}(-1)\,dw}{-w} \overset{x=w}{=} \int_{\rho}^{R} -\frac{e^{-ix}}{x}\,dx$$

$$I_3 = \int_{\rho}^{R} \frac{e^{ix}}{x}\,dx$$

$$\Rightarrow I_1 + I_3 = \int_{\rho}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx = 2i\int_{\rho}^{R} \frac{\sin x}{x}\,dx.$$

as $\rho \downarrow 0$ & $R \to \infty$:

$$\boxed{I_1 + I_3 \to 2i\int_0^\infty \frac{\sin x}{x}\,dx = 2iI}$$

Since f & C satisfy the cond's of Cauchy-Goursat,

$(+) \Rightarrow I = \int_0^\infty \frac{\sin x}{x}\,dx = -\frac{1}{2i}\left(\lim_{\rho\downarrow 0} I_2 + \lim_{R\to\infty} I_4\right),$

if these limits exist.

$$I_2 = \int_{C_\rho} \frac{e^{iz}}{z}\,dz \qquad z = \rho e^{i\theta} \quad \theta : \pi \to 0.$$

$$= \int_\pi^0 \frac{e^{i\rho e^{i\theta}}}{\cancel{\rho e^{i\theta}}}\, i\cancel{\rho e^{i\theta}}\,d\theta$$

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$$= -i\int_0^\pi e^{i\rho e^{i\theta}}\,d\theta$$

So, since $|\rho e^{i\theta}| = \rho$, $e^{i\rho e^{i\theta}} \to 1$ as $\rho \to 0$ uniformly in θ for $\theta \in [0,\pi]$.

$$\Rightarrow \lim_{\rho\downarrow 0} I_2 = -i\int_0^\pi \lim_{\rho\downarrow 0}\left(e^{i\rho e^{i\theta}}\right)d\theta$$

$$\Rightarrow I_2 \to -i\int_0^\pi 1\,d\theta = -i\pi \text{ as } \rho\downarrow 0.$$

Also, $I_4 \to 0$ as $R \to \infty$ by Jordan's Lemma (proof: next)

$$(+) \Rightarrow I = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{-1}{2i}\left(-i\pi + 0\right)$$

$$= \pi/2 \qquad \square$$

Lemma (Jordan) Let f be analytic on $\{z : \mathrm{Im}(z) \geq 0\} \cap \{z : |z| \geq R_0\}$, & satisfy

$$|f(z)| \leq M/R^{\beta} \quad (M, \beta > 0 \text{ constants}) \qquad (*)$$

on $\Gamma_R = \{Re^{i\theta},\ R > R_0,\ 0 \leq \theta \leq \pi\}$.

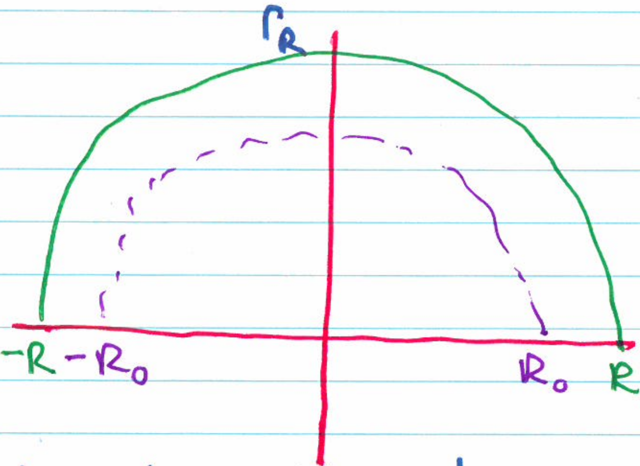
Then $\displaystyle\lim_{R\to\infty} \int_{\Gamma_R} e^{i\alpha z} f(z)\, dz = 0$ for any fixed $\alpha > 0$.

RMK: This applies to I_4 from Ex2, with: $f(z) = 1/z$ ($\Rightarrow M=1, \beta=1$) & $\alpha = 1$.

Pf:

on Γ_R, $z = Re^{i\theta}$

$dz = iRe^{i\theta}\, d\theta$



$$\left|\int_{\Gamma_R} e^{i\alpha z} f(z)\,dz\right| = \left|\int_0^{\pi} e^{i\alpha Re^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\, d\theta\right|$$

$$\leq R\int_0^{\pi} \left|e^{i\alpha Re^{i\theta}} f(Re^{i\theta})\right| d\theta$$

$$= R\int_0^{\pi} \left|e^{i\alpha(R\cos\theta + iR\sin\theta)} f(Re^{i\theta})\right| d\theta$$

$$= R\int_0^{\pi} e^{-\alpha R\sin\theta} \left|f(Re^{i\theta})\right| d\theta$$

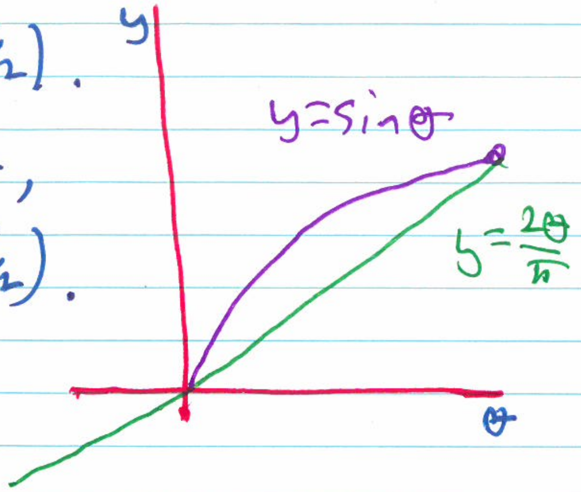
$$\overset{(*)}{\leq} \frac{M}{R^{\beta-1}} \int_0^{\pi} e^{-\alpha R\sin\theta}\, d\theta$$

$$= \frac{2M}{R^{\beta-1}} \int_0^{\pi/2} e^{-\alpha R\sin\theta}\, d\theta \qquad (**)$$

Note $\sin\theta > \frac{2\theta}{\pi}$ for $\theta \in (0, \pi/2)$.

(since $\sin\theta = 2\theta/\pi$ for $\theta = 0$ & $\pi/2$,

& $\left(\frac{2\theta}{\pi}\right)'' = 0$, $(\sin\theta)'' < 0$ on $(0, \pi/2)$.



So (**) $\leq \dfrac{2M}{R^{\beta-1}} \displaystyle\int_0^{\pi/2} e^{-\frac{2\alpha R}{\pi}\theta}\, d\theta$

$$= \frac{2M}{R^{\beta-1}} \cdot \frac{\pi}{2\alpha R}\left(1 - e^{-\alpha R}\right)$$

$\to 0$ as $R \to \infty$. $\square$

Next Rouché's thm.