LECTURE 32

§74 (8Ed §68) Singularities

$f : \Omega \to \mathbb{C}$ has a singularity at z_0 if :

* f is not analytic at z_0;
* Given any $\varepsilon > 0$, $\exists z_1 \in B_\varepsilon(z_0)$ s.t. f is analytic at z_1.

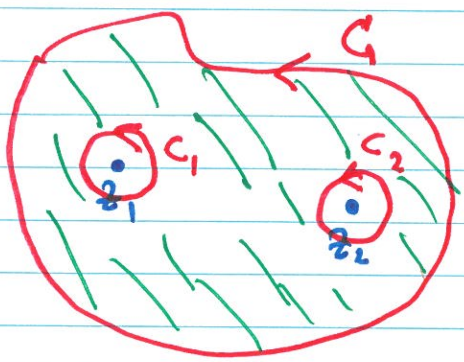
f has an isolated singularity at z_0 if f is analytic on $B_\varepsilon(z_0) \setminus \{z_0\}$ for some $\varepsilon > 0$.

Exs:

* $1/z$ has an isolated singularity at 0;
* $\operatorname{Log} z$ has non-isolated singularities on -ve Re axis $\cup \{0\}$;
* $\dfrac{\sin z}{z}$ has an isolated sing. at 0;
* $\dfrac{1}{\sin(\pi/z)}$ has $\begin{cases} \text{isolated sings at } z = 1/k,\ k \in \mathbb{Z} \setminus \{0\} \\ \text{nonisolated sing at } 0. \end{cases}$

(Cauchy) residue thm:

Suppose C is a +ve oriented simple closed contour, & that f is analytic on
$$C \cup \{\operatorname{Int} C \setminus \{z_1, \dots, z_k\}\}$$
Then
$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{res}_{z=z_j} f(z_j)}$$



Pf: Take disjoint $C_1, \dots, C_k$ disjoint, +vely oriented circles centred at $z_1, \dots, z_k$, with disjoint interiors, all lying in Int C.
Then $C, C_1, \dots, C_k$ form the bdy of a multiply-connected domain, called Ω. Then f is analytic on Ω & $\partial\Omega$, so Cauchy-Goursat extension
$\Rightarrow$
$$\int_C f(z)\,dz = \sum_{j=1}^{k} \int_{C_j} f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{res}_{z=z_j} f(z),$$
cf. def^n of b_1 in Lecture 31.

Classifying isolated Sings (§78, 8Ed §72)

If z_0 is an isolated sing. of f, then $\exists R>0$ s.t. f has a Laurent series expansion on $B_R(z_0)\setminus\{z_0\}$:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}.$$

CASE I : $b_n = 0 \ \forall n$.

Singularity is removable : setting $f(z_0)=a_0$ would extend f to be analytic on $B_R(z_0)$.

CASE II : at least one, but only finitely many of the b_n's are nonzero. Such a singularity is called a pole.

Define $m = \max\{n : b_n \neq 0\}$. m is called the order of the pole.

$m=1 \iff$ simple pole.

CASE III : ∞ly many of the b_n's $\neq 0$: z_0 is called an essential singularity.

Exs of classifying isolated sings:

1. $f(z) = \frac{\sin z}{z}$ analytic on $\mathbb{C}_*$ (quotient of entire f"s, denom $= 0$ only at 0).

$z = 0$ is an isolated sing which is removable:

$$g(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

is entire, & $g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$

$$\operatorname*{res}_{z=0} g(z) = \operatorname*{res}_{z=0} f(z) = 0.$$

2. $f(z) = 1/z^4$: analytic on $\mathbb{C}_*$, isolated sing at 0, which is a pole of order 4 (largest -ve power occuring in the Laurent expansion of f, which here is just f).

$\operatorname*{res}_{z=0} f(z) =$ coeff of z^{-1} in Laurent series $= 0$.

3. $h(z) = \frac{\sinh z}{z^4}$: analytic on $\mathbb{C}_*$ (quotient of analytic f"s, denom $= 0$ only at 0).

Isolated sing at 0.

On $h(z) = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$

$$= \left(\frac{1}{z^3} + \frac{1}{3!\,z} + \frac{z}{5!} + \cdots\right)$$

So h has a pole of order 3 (NOT 4) at 0, &

$$\operatorname*{res}_{z=0} h(z) = \frac{1}{3!} = \frac{1}{6}.$$

RESIDUE at a pole of order m (§80, 8Ed §73).

THM 1 An isolated sing z_0 of f is a pole of order $m \geq 1$ iff f can be written near z_0 as

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \quad \text{where:}$$

$$\begin{cases} \phi \text{ is analytic on some } B_\varepsilon(z_0) \\ \phi(z_0) \neq 0. \end{cases}$$

$(m-1)$th derivative

THM 2 In this case, $\boxed{\operatorname*{res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}}$

In particular, for $m=1$ (simple pole):

$$\operatorname*{res}_{z=z_0} f(z) = \frac{\phi^{(0)}(z_0)}{0!} = \phi(z_0)$$