LECTURE 33

§92 (8Ed §85) Integrals o.t.f. $\int_0^{2\pi} R(\sin t, \cos t)\,dt$

Try the subsitution $z = \cos t + i \sin t$.

$z^{-1} = \cos t - i \sin t$.

motivation $z = e^{it}$ $0 \le t \le 2\pi$.

$\Rightarrow$ (check!) $\cos t = \frac{1}{2}(z + z^{-1})$;

$\sin t = \frac{1}{2i}(z - z^{-1})$

$dz = (-\sin t + i \cos t)\,dt = iz\,dt$.

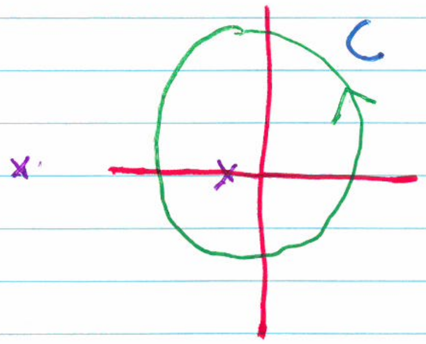
E.g. find $I = \int_0^{2\pi} \frac{dt}{2 + \cos t}$

Put $z = e^{it}$: by above, $dt = dz/iz$ &

$\cos t = \frac{1}{2}(z + z^{-1})$.

So for C given by $\{z = e^{it},\ 0 \le t \le 2\pi\}$, we have:

$$I = \int_C \frac{dz/iz}{2 + \frac{1}{2}(z + z^{-1})}$$



$$= -i \int_C \frac{dz}{2z + \frac{1}{2}(z^2 + 1)}$$

$$= -2i \int_C \frac{dz}{z^2 + 4z + 1} \quad (*)$$

To evaluate the integral in (*); note that the integrand is analytic except at the zeros of the denominator, i.e. $-2 \pm \sqrt{3}$. Only $-2 + \sqrt{3}$ is inside C, so by the Res Thm & (*) :

$$I = (-2i)2\pi i \operatorname*{res}_{z = -2 + \sqrt{3}} \frac{1}{z^2 + 4z + 1} \quad (\ast\ast)$$

To calculate the res, note that

$$\frac{1}{z^2+4z+1} = \frac{\phi(z)}{z-(-2+\sqrt{3})}, \text{ where}$$

$\phi(z) = \dfrac{1}{z-(-2-\sqrt{3})}$ is analytic & non-vanishing near $z=-2+\sqrt{3}$. Hence, $\dfrac{1}{z^2+4z+1}$ has a simple pole at $-2+\sqrt{3}$, & Thm. Lec 29

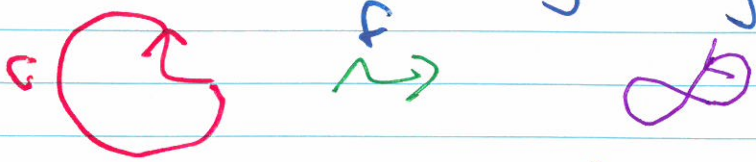
$$\Rightarrow \operatorname*{res}_{z=-2+\sqrt{3}} \frac{1}{z^2+4z+1} = \phi(-2+\sqrt{3}) = 1/2\sqrt{3}.$$

Hence (**) $\Rightarrow I = (-2i)(2\pi i)\cdot\dfrac{1}{2\sqrt{3}} = 2\pi/\sqrt{3}.$

Argument Principle & Rouché's Th^{m}:

§§93-94, 8th ed §§86-87.

f analytic in & on C, a +vely oriented simple closed contour, except possibly for poles inside C (note: finitely many poles).



Argument principle: $\Delta_C \arg f = 2\pi(Z - P)$

change in the argument of f as we go around C once

where Z = # zeros inside C, counting multiplicity,

& P = # poles inside C, counting total order.

E.g. $f(z) = \dfrac{z}{(z-\frac{1}{2})^3(z+\frac{i}{6})^5}$ satisfies:

$P=8$, $Z=1$ for C = unit circle, +vely oriented.

$P=5$, $Z=1$ for C = circle of radius $\frac{1}{4}$, centre 0, +vely oriented.

Application: Rouché's thm: Let f & g be analytic on C & in Int C, where C is a simple closed contour, orientation irrelevant.

Suppose $|f(z)| > |g(z)| \quad \forall z \in C$.

Then f & $f+g$ have the same # of zeros counting multiplicity inside C.

Note e.g., $f(z) = (z-i)^2(z+i)^3$ has 5 zeros in $\mathbb{C}$ counting multiplicity.

Example of Rouché:

How many zeros of $h(z) = z^7 - 4z^3 + z - 1$ lie inside the unit circle?

Sol^n: Put $f(z) = -4z^3$

$$g(z) = z^7 + z - 1.$$

f & g are entire (polys).

$|f| = 4$ on C, since $|z| = 1$ on C.

Note
$$\begin{aligned} |g(z)| &= |z^7 + z - 1| \\ &\le |z^7| + |z| + |-1| \\ &= 3 < 4 = |f(z)| \text{ on } C. \end{aligned}$$

So Rouché $\Rightarrow$ f & $f+g = h$ have the same # zeros (counting multiplicity) inside C, namely 3, since f has a 3-fold zero at 0.