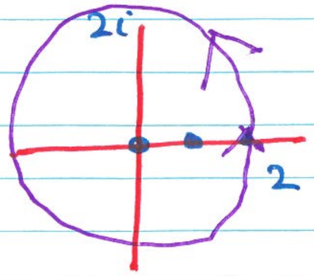
Lecture 33

Contour Integrals & Residues.

Aim: calculate $I = \int_C \underbrace{\frac{5z-2}{z(z-1)}}_{f} dz$



Note f is analytic on $\mathbb{C} \setminus \{0,1\}$ (rational f^n).

METHOD 1:

Look at $0<|z|<1$.

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{1}{1-z}(-1)\left(\frac{5z-2}{z}\right)$$

$$= (1+z+z^2+\dots)(5-\tfrac{2}{z})(-1)$$

$$= (1+z+z^2+\dots)(\tfrac{2}{z}-5).$$

$$= \frac{2}{z} + (-5+2) + (2-5)z + (2-5)z^2 + \dots$$

$$= \frac{2}{z} - 3 - 3z - 3z^2 + \dots.$$

$\Rightarrow \operatorname{res}_{z=0} f(z) = \text{coeff of } \frac{1}{z} = 2.$

Now look at 1: $0<|z-1|<1$

$$f(z) = \frac{5(z-1)+3}{(z-1)} \cdot \frac{1}{(z-1)+1}$$

$$= \left(5 + \frac{3}{z-1}\right)\left(\frac{1}{1-(-(z-1))}\right)$$

$$= \left(5+\frac{3}{z-1}\right)\left(1 + (-(z-1)) + (z-1)^2 + \dots\right)$$

$\Rightarrow \operatorname{res}_{z=1} f(z) = \text{coeff of } \frac{1}{z-1} = 3.$

Cauchy res thm $\Rightarrow I = 2\pi i(2+3) = 10\pi i.$

METHOD 2

Note: $f(z) = \dfrac{3}{z-1} + \dfrac{2}{z}$

$2/z$ is analytic on $\mathbb{C} \setminus \{0\}$

$3/(z-1)$ is analytic on $\mathbb{C} \setminus \{1\}$.

Therefore $\operatorname{res}_{z=0} f(z) = 2$, $\operatorname{res}_{z=1} f(z) = 3$, so

res Thm $\Rightarrow$ $I = 10\pi i$.

METHOD 3

near $z=1$, $f(z) = \dfrac{\phi(z)}{z-1}$, for $\phi(z) = \dfrac{5z-2}{z}$

ϕ is analytic, non-zero.

So, Thm 1 Lec 32 $\Rightarrow$ simple pole at 1, &

$\operatorname{res}_{z=1} f(z) = \phi(1)$ via Thm 2 Lec 32.

$= \dfrac{5-2}{1} = 3$.

near $z=0$

Another example of calculating residues:

$g(z) = \dfrac{z^3+2z}{(z-i)^3}$: find & classify sings, find residues.

Analytic on $\mathbb{C} \setminus \{i\}$ (rational f^n). i is an isolated sing.

Near $z=i$, we have $g(z) = \phi(z)/(z-i)^3$

Where $\phi(z) = z^3+2z$ is entire & $\phi(i) = i \neq 0$.

Thm 1 Lecture 32 $\Rightarrow$ pole of order 3 at i,

& Thm 2 Lec 32 $\Rightarrow$ $\operatorname{res}_{z=i} g(z) = \dfrac{\phi^{(3-1)}(i)}{(3-1)!} = 3i$.

§82-83 Zeros & Poles (8 Ed §75-76)

f analytic at z_0 : f has a zero of order m at z_0 if
$$\begin{cases} f^{(j)}(z_0) = 0 & j = 0, \dots, m-1 \\ f^{(m)}(z_0) \neq 0. \end{cases}$$

E.g. $f(z) = (z-i)^4(z-4)$ has a zero of order 4 at i & a zero of order 1 (a.k.a. a simple zero) at $z=4$.

§82 Th^m: 1 f analytic at z_0 has a zero of order m at $z_0 \Leftrightarrow f(z) = (z-z_0)^m g(z)$, where g is analytic at z_0, $g(z_0) \neq 0$.

§83 Th^m: 1 Suppose p & q are analytic at z_0, $p(z_0) \neq 0$, q has a zero of order m at z_0. Then p/q has a pole of order m at z_0.

Ex: $p(z) = 1$, $q(z) = z(e^z - 1)$.

$\frac{p}{q}$ has isolated sing at zero. (Quotient of analytic f^{n}s, sing where the denom vanishes)

q is entire
$$q(0) = 0$$
$$q'(0) = 0$$
$$q''(0) = 2 \neq 0.$$

Th^m: $\Rightarrow$ p/q has a pole of order 2 at 0.

THM 2: Let p, q be analytic at z_0.
If $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$.
Then p/q has a simple pole at z_0 via Th^m 1,
& $\operatorname*{res}_{z=z_0} \frac{p}{q}(z) = \frac{p(z_0)}{q'(z_0)}$

These are higher-order analogues, but the are messy.