LECTURE 34

Classifying isolated sings, calculating residues.

Ex 1 $f(z) = \dfrac{z+i}{z^2+9}$.

f is analytic on $\mathbb{C} \setminus \{\pm 3i\}$ (rational f^{n})

$\pm 3i$ are isolated singularities.

Near $z = 3i$, write $f(z) = \dfrac{\phi(z)}{z-3i}$,

where $\phi(z) = \dfrac{z+i}{z+3i}$ is analytic &

non-zero near $z=3i$ $\left(\phi(3i) = 4i/6i \neq 0\right)$

Th^{m} 1, Lec 33 $\Rightarrow$ simple pole at $3i$,

Th^{m} 2 (*) Lec 33 $\Rightarrow$ $\operatorname*{res}_{z=3i} f(z) = \phi(3i) = 2/3$.

Near $z = -3i$, ... ☆

Ex 2 $g(z) = \dfrac{z^3+2z}{(z-i)^3}$

Analytic on $\mathbb{C} \setminus \{i\}$ (rational f^{n})

Near $z=i$, we have $g(z) = \phi(z)/(z-i)^3$

where $\phi(z) = z^3 + 2z$ is entire & $\phi(i) = i \neq 0$.

Th^{m} 1, Lec 33 $\Rightarrow$ pole of order 3 at i, &

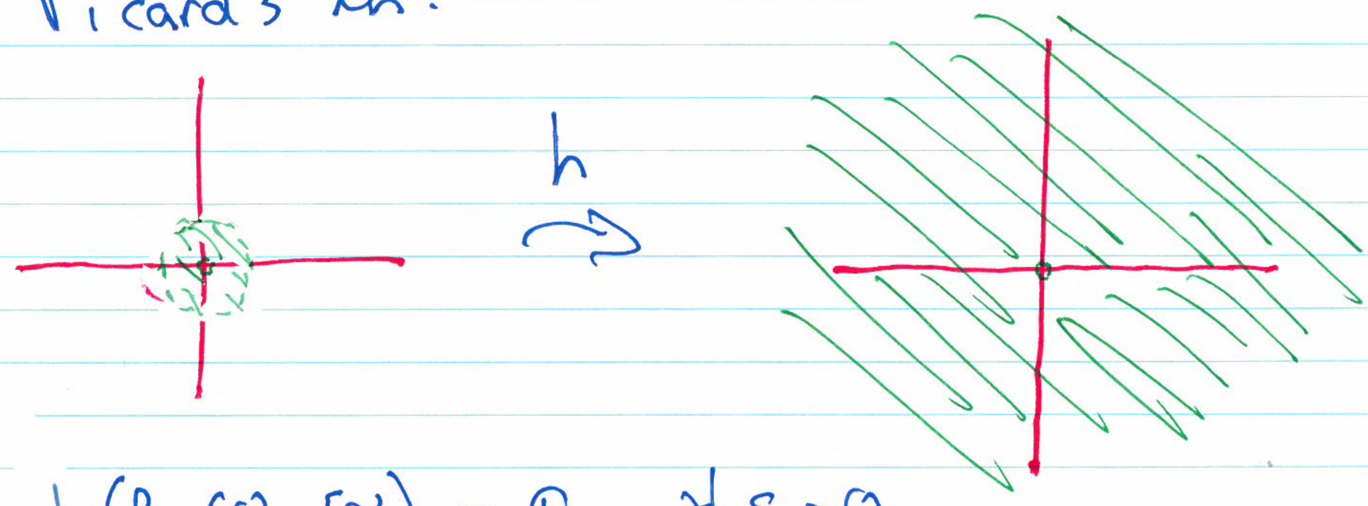
(*) $\Rightarrow$ $\operatorname*{res}_{z=i} g(z) = \dfrac{\phi^{(3-1)}(i)}{(3-1)!} = 3i$.

Ex3

$h(z) = e^{1/z}$: analytic on $\mathbb{C}_*$, isolated sing at 0, $\operatorname{res}_{z=0} h(z) = 1$, noting

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n! z^n} \quad \text{on } \mathbb{C}_*.$$

Picard's th^m:



$h(B_\varepsilon(0) \setminus \{0\}) = \mathbb{C}_* \quad \forall \varepsilon > 0$

(note: no sol^{n}s to $e^{1/z} = 0$).

Ex4 $\gamma(z) = \cfrac{1}{1 - 1/z}$ analytic on $\mathbb{C} \setminus \{0,1\}$ (composition of rational f^{n}s), isolated sings at $0, 1$. For $z \neq 0, 1$:

$$\gamma(z) = \frac{z}{z-1} = \frac{-z}{1-z}$$

$$= -z(1 + z + z^2 + \dots) \qquad 0 < |z| < 1$$

$$= -z - z^2 - z^3 - \dots$$

$\Rightarrow$ removable sing at 0 (no b's in the Laurent series): set $\gamma(0) = 0$ extends γ to an analytic f^n on $\{z : |z| < 1\}$.

$2 = 1$ ☆

§82–83 Zeroes & poles (8Ed §75–76)

f analytic at z_0: f has a zero of order m at z_0 if
$$\begin{cases} f^{(j)}(z_0)=0 & j=0,\dots,m-1. \\ f^{(m)}(z_0)\neq 0 \end{cases}$$

E.g. $f(z)=(z-i)^4(z-4)$ has a zero of order 4 at $z=i$ & a zero of order 1 (a.k.a. a simple zero) at $z=4$.

§82 Th^m 1 f analytic at z_0 has a zero of order m at z_0 $\iff$ $f(z)=(z-z_0)^m g(z)$, where g is analytic at z_0, & $g(z_0)\neq 0$.

§83 Th^m. 1 Suppose p & q are analytic at z_0, $p(z_0)\neq 0$, q has a zero of order m at z_0. Then p/q has a pole of order m at z_0.

Ex $p(z)=1$, $q(z)=z(e^z-1)$.
p/q has an isolated sing at 0 (quotient of analytic f^n's). p is analytic and non-zero on $\mathbb{C}$. q is entire, &
$$q(0)=0$$
$$q'(0)=0$$
$$q''(0)=2\neq 0.$$

Th^m. $\Rightarrow$ p/q has a pole of order 2 at 0.

THM 2 Let p, q be analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$, then p/q has a simple pole at z_0 via Th^m 1, and

$$\operatorname*{res}_{z=z_0} \frac{p}{q}(z) = \frac{p(z_0)}{q'(z_0)}$$

There are higher-order analogues, but they're messy.

Example

Consider $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

$p(z) = \cos z$, $q(z) = \sin z$ are entire, so sings of f are zeros of $\sin$, i.e., $n\pi$ $n \in \mathbb{Z}$.

Note
$$p(n\pi) = (-1)^n \quad n \in \mathbb{Z}.$$
$$q(n\pi) = 0$$
$$q'(n\pi) = \cos(n\pi) = (-1)^n \neq 0.$$

So Th^m 2 $\Rightarrow$ each sing $z_n = n\pi$ of f is a simple pole, with

$$\operatorname*{res}_{z=z_n} f(z) = \frac{p(z_n)}{q'(z_n)} = \frac{(-1)^n}{(-1)^n} = 1.$$

revisit exs from Lec 33 with these techniques.

Next: contour integrals.