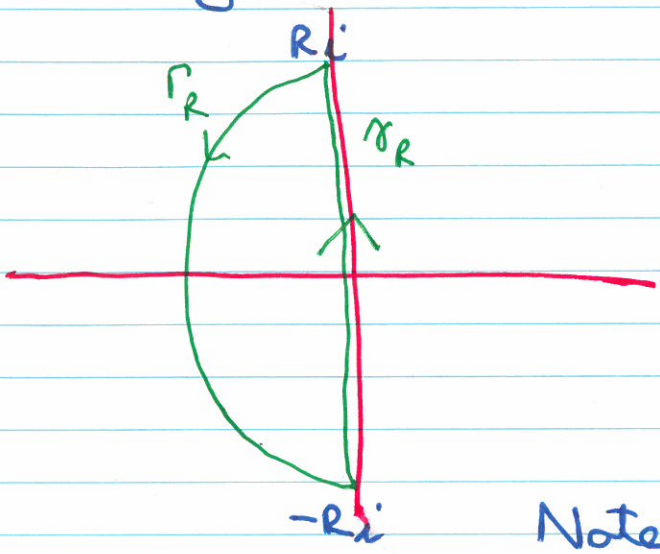
LECTURE 34

Rouché Ex 2:

2020 exam, Q7 Show $h(z) = 2z+4+3e^{z}$ has precisely one zero in the Left Half Plane.



Sol^{n}: consider the contour $C_R = \gamma_R + \Gamma_R$, for R to be determined later.

Put $f(z) = 2z+4$, $g(z) = 3e^{z}$.

Note f & g are entire (poly, exp). On γ_R, $z = iy$, so $|f(z)| = |4+2iy| \geq 4$ & $|g(z)| = 3|e^{iy}| = 3 < 4 = |f(z)|$.

On Γ_R, $z = x+iy$ $x \leq 0$, $|z| = R$.

$\Rightarrow |g(z)| = 3|e^{x+iy}| = 3e^{x} \leq 3$ since $x \leq 0$,

$$\begin{aligned} \& \ |f(z)| = |2z+4| &\geq |2|z|-4| \\ &= |2R-4| \\ &= 2R-4 \quad \text{for } R \geq 2 \\ &> 3 \quad \text{for } R > 7/2. \end{aligned}$$

So for $R > 7/2$, $|f| > |g|$ on Γ_R & hence on C_R.

Rouché $\Rightarrow$ f & $f+g = h$ have the same # zeros inside C_R. f has precisely one zero (at $z=-2$), so h does as well. Letting $R \to \infty$ $\Rightarrow$ h has precisely one zero in the LHP.

Final example for Laurent series.

Q^n: Laurent series for $\frac{1}{\sinh z}$ $= f(z)$ at 0?

Ans: f is $1/$entire fn, so is analytic except for the zeros of denominator, i.e., $z = n\pi i$, $n \in \mathbb{Z}$. So 0 is an isolated singularity of f, & f has a Laurent series on $0<|z|<\pi$.

$$f(z) = \frac{1}{\sinh z} = \frac{1}{z + z^3/3! + z^5/5! + \cdots}$$

$$= \frac{1}{z} \cdot \frac{1}{1 + z^2/3! + z^4/5! + \cdots}$$

$$= \frac{1}{z} \cdot \frac{1}{1 - (-z^2/3! - z^4/5! - \cdots)} \quad (*)$$

Using the formula for the sum of a geometric series, for $|z|$ small enough, i.e., s.t. $|z^2/3! + z^4/5! + \cdots| < 1$, there holds:

$$(*) = \frac{1}{z}\left[1 + (-z^2/3! - z^4/5! - \cdots) + (-z^2/3! - z^4/5! - \cdots)^2 + \cdots\right]$$

$$= \frac{1}{z} - \frac{z}{6} + \frac{7z^3}{360} + \cdots$$

In particular, simple pole at 0, with

$$\operatorname*{res}_{z=0} \frac{1}{\sinh z} = 1 \quad (\text{coeff. of } 1/z)$$

Note: can also do by division of power series.

Lecture 18: review covering up to mid-sem.

* line integrals
* contour-integrals
* Cauchy-Goursat, Cauchy's Integral Formula.
* Formulae for derivatives; analytic $\Rightarrow$ ∞ly diff^ble.
* Morera's thm.
* Liouville – Fundamental thm of algebra.
* Conformal mappings
* Bdy value problems
* Harmonic functions
* Harmonic conjugate
* Transformation of bvps
* Power series / Taylor Series / Laurent Series.
* analytic $\Leftrightarrow$ equal to Taylor series
* Singularities isolated / non-isolated
* Classifying isolated sings
  * removable
  * pole (order)
  * essential
* Residue, methods for calculating.
* Integration on $\mathbb{R}$ using contour integrals & the residue thm, properties of PV-integrals.
* M-L thm, Jordan's Lemma.

Arg principle $\rightsquigarrow$ Rouché's Thm. not directly examinable.

* Verify cond^ns of thm's, definitions.
* Watch your algebra, sanity check.
* Remember you are in $\mathbb{C}$.