LECTURE 34

Example for Th^{m} 2, Lec 33:

Consider $f(z) = \cot z = \dfrac{\cos z}{\sin z}$.

$p(z) = \cos z$, $q(z) = \sin z$ are entire f^{n}s,
so sings of f are zeros of sin, i.e., $n\pi$ $n \in \mathbb{Z}$.

$p(n\pi) = (-1)^n \quad n \in \mathbb{Z}$.

$q(n\pi) = 0$

$q'(n\pi) = \cos(n\pi) = (-1)^n \neq 0$.

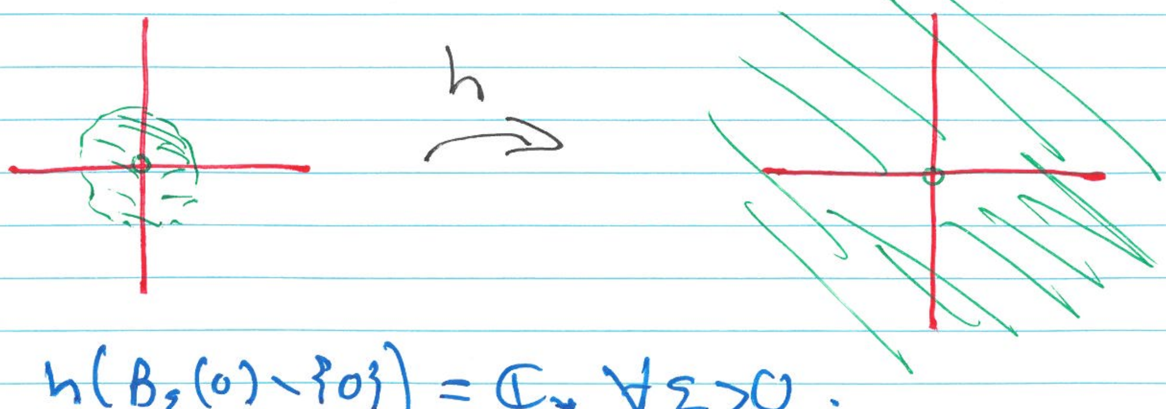
So Th^{m} 2, Lec 33 $\Rightarrow$ each singularity $z_n = n\pi$ of f is a simple pole, with

$$\operatorname*{res}_{z=z_n} f(z) = \frac{p(z_n)}{q'(z_n)} = \frac{(-1)^n}{(-1)^n} = 1.$$

Suggestion: revisit some of the earlier exs.

Example 1 $\quad h(z) = e^{1/z}$: analytic on $\mathbb{C}_*$, isolated sing at 0, $\operatorname*{res}_{z=0} h(z) = 1$, noting

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n! z^n} \text{ on } \mathbb{C}_*.$$



$h(B_\varepsilon(0) \setminus \{0\}) = \mathbb{C}_* \ \forall \varepsilon > 0$.

Note: no sol^{n}s to $e^{1/z} = 0$.

Example of
<u>Picard's (great) Theorem</u>: if g has an ess. sing at z_0, the g maps any deleted nbhd of z_0 onto all of $\mathbb{C}$, or all of $\mathbb{C} \setminus \{1 \text{ point}\}$.

<u>Ex 2</u> $\quad \gamma(z) = \dfrac{1}{1 - 1/z}$ analytic on $\mathbb{C} \setminus \{0, 1\}$ (composition of rational f^ns)

For $z \neq 0, 1$:

$$\gamma(z) = \frac{z}{z-1} = \frac{-z}{1-z}$$

$$= (-z)(1 + z + z^2 + \dots) \quad 0 < |z| < 1$$

$$= -z - z^2 - z^3 - \dots$$

$\Rightarrow$ removable sing at 0 (no b's in the Laurent series): set $\gamma(0) = 0$ extends γ to an analytic f^n on $\{z : |z| < 1\}$.

<u>$z = 1$</u>: $\star$

Key application of Res Th^{m}:

Use contour integrals in $\mathbb{C}$ to evaluate integrals over $\mathbb{R}$: typically from $-\infty$ to ∞ or from 0 to ∞.

Recall: $(*)$ $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{M_1\to-\infty} \int_{M_1}^{0} f(x) + \lim_{M_2\to\infty} \int_0^{M_2} f(x)\,dx$

if both the limits on the RHS exist.

Rmk: you can replace 0 by any fixed $c \in \mathbb{R}$.

You cannot in general replace RHS of $(*)$ by
$$\lim_{M\to\infty} \int_{-M}^{M} f(x)\,dx.$$

If you do this anyway, it defines the Cauchy Principal Value (PV) Integral.

E.g. $\int_{-\infty}^{\infty} x\,dx = \lim_{M_1\to-\infty} \int_{M_1}^{0} x\,dx + \lim_{M_2\to\infty} \int_0^{M_2} x\,dx$

$$= \lim_{M_1\to-\infty} \frac{-M_1^2}{2} + \lim_{M_2\to\infty} \frac{M_2^2}{2},$$

which is undefined.

However, $PV\int_{-\infty}^{\infty} x\,dx = \lim_{M\to\infty} \int_{-M}^{M} x\,dx$

$$= \lim_{M\to\infty} \left[\frac{M^2}{2} - \frac{(-M)^2}{2}\right]$$

$$= \lim_{M\to\infty} 0 = 0.$$

E.g. for f even, or for $f \geq 0$.

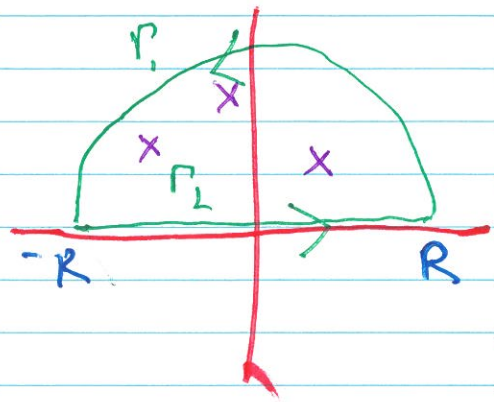
In particular, for f even, i.e., $f(x) = f(-x)\ \forall x \in \mathbb{R}$, we have $\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\,dx = \frac{1}{2}\,PV\int_{-\infty}^\infty f(x)\,dx$,

& all of these converge or diverge together.

How does this fit with contour integrals?

Suppose f is even & "nice" on $\mathbb{R}$.

Aim: evaluate $\int_{-\infty}^\infty f$.



Suppose f is analytic inside & on the contour $C = \Gamma_1 + \Gamma_2$, except possibly for isolated singularities in Int C.

These holds: $\int_C f = \underbrace{\int_{\Gamma_1} f}_{\mathrm{I}} + \underbrace{\int_{\Gamma_2} f}_{\mathrm{II}}$.

LHS: hopefully can evaluate via Res Th^{m}.

Let $R \to \infty$: $\mathrm{II} \xrightarrow[R\to\infty]{} PV\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^\infty f(x)\,dx$,

since f is even.

Only remains to deal with $\lim_{R\to\infty}\int_{\Gamma_1} f$: hopefully $= 0$, e.g. via M-ℓ estimate.

$f(x)$

Example: evaluate $I = \int_0^\infty \frac{x^2}{1+x^6}\,dx$

Note f is cts & even on $\mathbb{R}$, $f \sim 1/x^4$ as $x \to \pm\infty$

So I converges (p-test, $p>1$).

Note $f(z) = \dfrac{z^2}{1+z^6}$ is rational f^n, & is analytic except for the 6 zeros of the denominator, i.e., sol^{n}s of $z^6+1=0$.