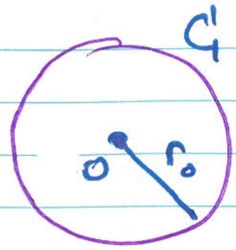
LECTURE 34 a).

Poisson's integral formula §135 (8Ed §124).



Recall Cauchy: if f analytic in & on C_0, then for $z \in \operatorname{Int} C_0$:

$$\boxed{f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(\zeta)}{\zeta - z}\,d\zeta} \quad (1)$$

Recall for $z = re^{i\theta}$, $r > 0$, the inverse pt to z_0 relative to the circle C_0 is

$r^* e^{i\theta}$ with $r^* r = r_0^2$

Recall $(z^*)^* = z$

Note $z^* = r^* e^{i\theta} = \dfrac{r_0^2}{r} e^{i\theta}$

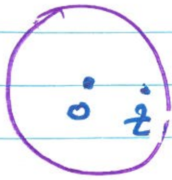
$= \dfrac{r_0^2}{re^{-i\theta}}$

$= \dfrac{r_0^2}{\bar{z}}$

$= \zeta\bar{\zeta}/\bar{z}$ for <u>any</u> $\zeta \in C_0$ (2)

Now fix $z \in \operatorname{Int} C_0$, $z_0 \neq 0$.

Note $\displaystyle\int_{C_0} \frac{f(\zeta)}{\zeta - z^*}\,d\zeta = 0$ since $\zeta \mapsto \dfrac{f(\zeta)}{\zeta - z^*}$ is analytic in & on C_0, since $z^* \in \operatorname{Ext} C_0$: follows by Cauchy-Goursat, Lec 22.



Combining ① & ② :

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \underbrace{\left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*} \right)}_{I} f(\xi)\, d\xi .$$

$$I = \left(\frac{\xi}{\xi - z} - \frac{\xi}{\xi - z^*} \right) \frac{1}{\xi} \qquad \frac{\xi\bar{\xi}}{\bar{z}}$$

$$= \left(\frac{\xi}{\xi - z} - \frac{1}{1 - \bar{\xi}/\bar{z}} \right) \frac{1}{\xi}$$

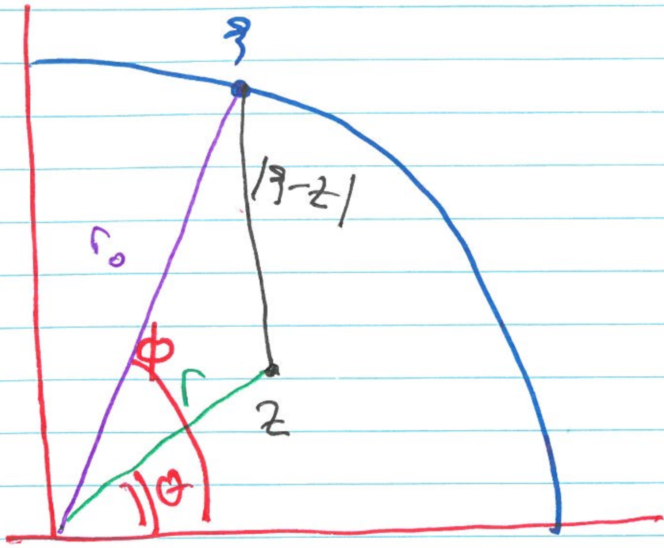
$$= \left(\frac{\xi}{\xi - z} - \frac{\bar{z}}{\bar{z} - \bar{\xi}} \right) \frac{1}{\xi}$$

$$= \frac{\xi\bar{\xi} - z\bar{z}}{|\xi - z|^2} \cdot \frac{1}{\xi} \qquad ③$$

$z = re^{i\theta}$ put $\xi = r_0 e^{i\phi}$, $0 \le \phi \le 2\pi$

$d\xi = r_0 i e^{i\phi} d\phi$ &

$$③ \Rightarrow I = \frac{(r_0^2 - r^2)}{|\xi - z|^2} \cdot \frac{1}{r_0 e^{i\phi}} \qquad ④$$



cos rule:

$$|\xi - z|^2 = r_0^2 + r^2 - 2r_0 r\cos(\phi-\theta) \qquad (5)$$

$$\Rightarrow f(re^{i\theta}) = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{r_0^2-r^2}{|\xi-z|^2}\cdot\cancel{\frac{1}{r_0e^{i\phi}}}\, f(r_0e^{i\phi})\cdot \cancel{r_0 i e^{i\phi}}\, d\phi,$$

$$\overset{(5)}{=} \frac{r_0^2-r^2}{2\pi}\int_0^{2\pi} \frac{f(r_0e^{i\phi})\,d\phi}{r_0^2+r^2-2r_0 r\cos(\phi-\theta)} \qquad (6)$$

Taking the real part of (6) $\Rightarrow$

Given "nice enough" $\Phi(r_0,\phi)$ defined on the bdy C_{r_0} of $B_{r_0}(0)$, a (in fact, the) sol^n of the Dirichlet problem

$$(D)\quad \begin{cases} \Delta u = 0 & \text{in } B_{r_0} \\ u|_{\partial B_{r_0}} = \Phi(r_0,\phi) \end{cases}$$

is given by:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{r_0^2-r^2}{r_0^2-2r_0 r\cos(\phi-\theta)+r^2}}_{\text{Poisson kernel } P(r_0,r,\phi-\theta)\ (\text{Poisson, }1885)} \Phi(r_0,\phi)\,d\phi$$

<u>Remark:</u> Valid for $z=0$. ☆

<u>Remark 2:</u> Does $\lim_{\substack{(r,\theta)\to(r_0,\theta_0) \\ r<r_0}} u(r,\theta) = \Phi(r_0,\theta_0)$?

(cf. BC $(r,\theta_0)\to(r_0,\theta_0)$ $r<r_0$).

Note: only makes sense when Φ is cts at (r_0,θ_0). At such points, the answer is "yes".

<u>Poisson kernel:</u>

* $P>0$ on B_{r_0}.
* $P(r_0,r,\phi-\theta) = Re\left(\frac{\zeta+z}{\zeta-z}\right)$
* $P(r_0,r,\phi-\theta)$ is a harmonic f^n of r & θ interior to C_0 for each fixed $\zeta\in C_0$.
* $P(r_0,r,\phi-\theta)$ is even in $\phi-\theta$, with period 2π.
* $P(r_0,0,\phi-\theta) = \frac{r_0^2}{r_0^2} = 1$.
* $\int_0^{2\pi} P(r_0,r,\phi-\theta)\,d\phi = 2\pi$