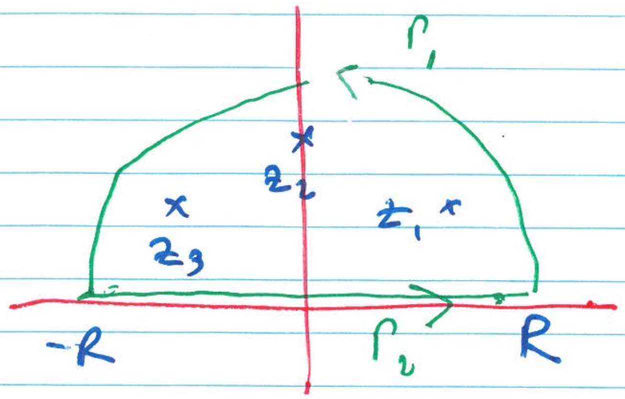
$$I = \int_0^\infty \frac{x^2}{1+x^6}\,dx$$

$f(x)$



$R>1$: f is analytic in & on $C = \Gamma_1 + \Gamma_2$, except for the 3 zeros of z^6+1 in the UHP:

$$z_1 = e^{\pi i/6}, \quad z_2 = i, \quad z_3 = e^{5\pi i/6}$$

Res Th^{m}: $\Rightarrow \int_C f = 2\pi i \sum_{j=1}^{3} \operatorname{res}_{z=z_j} f(z)$ (1)

Note f has the form p/q: for each z_j, $j=1,2,3$ there holds:

$$p(z_j) = z_j^2 \neq 0$$

$$q(z_j) = 1+z_j^6 = 0$$

$$q'(z_j) = 6z_j^5 \neq 0.$$

$\Rightarrow$ simple pole at each z_j.

So, Th^{m}: 2 Lec 33 $\Rightarrow$ RHS of (1) $= 2\pi i \sum_{j=1}^{3} \frac{p(z_j)}{q'(z_j)}$

$$= 2\pi i \sum_{j=1}^{3} z_j^2/6z_j^5$$

$$= 2\pi i \sum_{j=1}^{3} 1/6z_j^3$$

$$= 2\pi i \left(\tfrac{1}{6i} - \tfrac{1}{6i} + \tfrac{1}{6i}\right)$$

$$= \pi/3.$$

Note: $\int_C F = \int_{\Gamma_1} F + \int_{\Gamma_2} F$ (2)

$\qquad\qquad\qquad (\alpha) \quad (\beta)$

As $R \to \infty$, $(\beta) \to 2I$

Claim $(\alpha) \to 0$ as $R \to \infty$.

Note: $|(\alpha)| \leq M_R L_R$ (*)

where $L_R = \text{length}(\Gamma_1) = \pi R$ &

$$M_R = \max_{z \in \Gamma_1} |f(z)|$$

$$\underset{R>1}{\leq} \max_{|z|=R} \left| \frac{z^2}{1+z^6} \right|$$

$$\leq \frac{R^2}{R^6 - 1}$$ via reverse Δ-ineq, $|a+b| \geq \big||a|-|b|\big|$

So (*) $\Rightarrow |(\alpha)| \leq \dfrac{\pi R \cdot R^2}{R^6 - 1}$

$$= \frac{\pi}{R^3 - 1/R^3} \to 0 \quad \text{as } R \to \infty.$$

So $\lim_{R\to\infty}$ in (2) $\Rightarrow \pi/3 = 0 + 2I \Rightarrow I = \pi/6$. $\square$

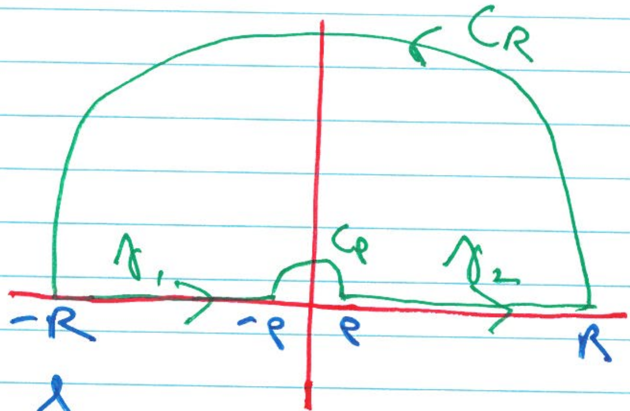
Ex 2 $I = \int_0^\infty \frac{\sin x}{x}\,dx$.

Note that $\frac{\sin x}{x}$ is even, so if I exists,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}PV\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx.$$

Take $f(z) = e^{iz}/z$.

$C = \gamma_1 + C_\rho + \gamma_2 + C_R$.



f is analytic on $\mathbb{C}_*$ & in particular on C & Int C, so $\int_C f = 0$ by Cauchy-Goursat.

$$\Rightarrow \underbrace{\int_{\gamma_1} f}_{I_1} + \underbrace{\int_{C_\rho} f}_{I_2} + \underbrace{\int_{\gamma_2} f}_{I_3} + \underbrace{\int_{C_R} f}_{I_4} = 0 \qquad (*)$$

$$I_1 = \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx \overset{w=-x}{=} -\int_{\rho}^{R} \frac{e^{-iw}(-1)\,dw}{-w} \overset{x=w}{=} \int_{\rho}^{R} \frac{-e^{-ix}\,dx}{x}$$

$$I_3 = \int_{\rho}^{R} \frac{e^{ix}}{x}\,dx$$

$$I_1 + I_3 = \int_{\rho}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx = 2i\int_{\rho}^{R} \frac{\sin x}{x}\,dx.$$

as $\rho \downarrow 0$ & $R \to \infty$:

$$\boxed{I_1 + I_3 \to 2i\int_0^\infty \frac{\sin x}{x}\,dx = 2iI}$$

$(*) \Rightarrow I = \int_0^\infty \frac{\sin x}{x}\,dx = -\frac{1}{2i}\left(\lim_{\rho\downarrow 0} I_2 + \lim_{R\to\infty} I_4\right)$, $(**)$

if these limits exist.

$I_2 = \int_{C_\rho} \frac{e^{iz}}{z}\,dz \qquad z = \rho e^{i\theta} : \theta : \pi \to 0.$

$= \int_\pi^0 \frac{e^{i\rho e^{i\theta}}}{\cancel{\rho e^{i\theta}}}\, i\cancel{\rho e^{i\theta}}\,d\theta$

$= -i\int_0^\pi e^{i\rho e^{i\theta}}\,d\theta$

Since $|\rho e^{i\theta}| = \rho$, $e^{i\rho e^{i\theta}} \to 1$ as $\rho \to 0$

uniformly in θ for $\theta \in [0,\pi]$.

$\Rightarrow \lim_{\rho\downarrow 0} I_2 = -i\int_0^\pi \lim_{\rho\downarrow 0}\left(e^{i\rho e^{i\theta}}\right)d\theta$

$\Rightarrow I_2 \to -i\int_0^\pi 1\,d\theta = -i\pi$ as $\rho \downarrow 0$.

Also, $I_4 \to 0$ as $R \to \infty$ by Jordan's Lemma

(proof : next)

So $(**) \Rightarrow I = \int_0^\infty \frac{\sin x}{x}\,dx = -\frac{1}{2i}(-i\pi + 0)$

$= \pi/2$. $\square$