LECTURE 36

Lemma (Jordan). Let f be analytic on $\{z : \operatorname{Im}(z) \geq 0\} \cap \{z : |z| \geq R_0\}$, & satisfy

$$|f(z)| \leq M/R^{\beta} \quad (M, \beta > 0 \text{ constants}), \; (*)$$

on $\Gamma_R = \{Re^{i\theta}, R > R_0, 0 \leq \theta \leq \pi\}$.

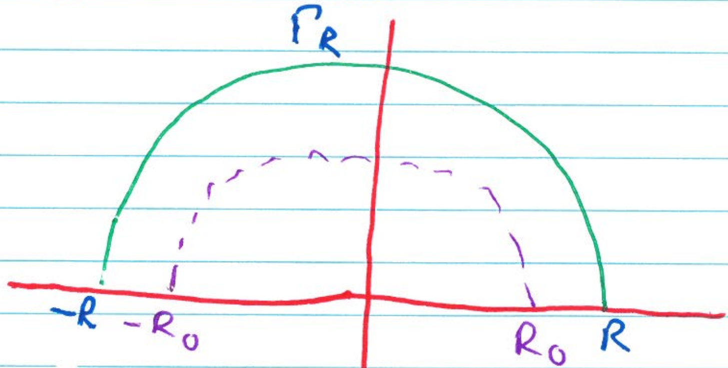
Then $\displaystyle\lim_{R \to \infty} \int_{\Gamma_R} e^{i\alpha z} f(z)\,dz = 0$ for any fixed $\alpha > 0$.

RMK: This applies to I_4 from the previous example with $f(z) = 1/z$ ($\Rightarrow M = 1, \beta = 1$) & $\alpha = 1$.

Pf:

On Γ_R, $z = Re^{i\theta}$

$dz = iRe^{i\theta}\,d\theta$.



$$\left|\int_{\Gamma_R} e^{i\alpha z} f(z)\,dz\right| = \left|\int_0^{\pi} e^{i\alpha Re^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta\right|$$

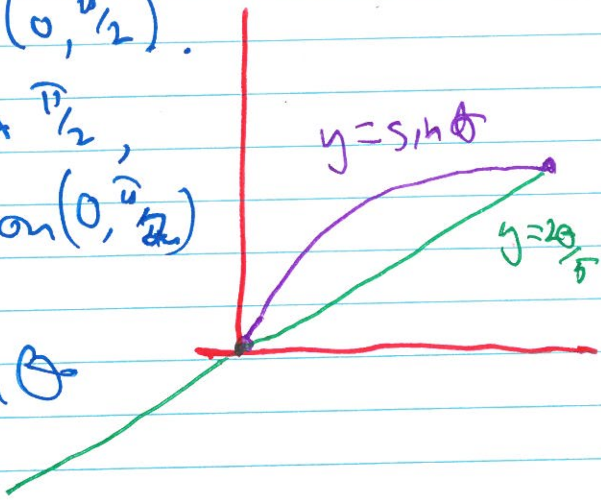
$$\leq R\int_0^{\pi} \left|e^{i\alpha Re^{i\theta}} f(Re^{i\theta})\right| d\theta$$

$$= R\int_0^{\pi} \left|e^{i\alpha(R\cos\theta + iR\sin\theta)} f(Re^{i\theta})\right| d\theta$$

$$\overset{(*)}{\leq} \frac{M}{R^{\beta-1}} \int_0^{\pi} e^{-\alpha R\sin\theta}\,d\theta.$$

$$= \frac{2M}{R^{\beta-1}} \int_0^{\pi/2} e^{-\alpha R\sin\theta}\,d\theta \quad (**)$$

Note: $\sin\theta > \frac{2\theta}{\pi}$ for $\theta \in (0, \pi/2)$.
(since $\sin\theta = \frac{2\theta}{\pi}$ for $\theta = 0$ & $\pi/2$,
& $\left(\frac{2\theta}{\pi}\right)'' = 0$, $(\sin\theta)'' < 0$ on $(0, \pi/2)$)



So $(**)$ $\leq \dfrac{2M}{R^{\beta-1}} \displaystyle\int_0^{\pi/2} e^{-\frac{2\alpha R\theta}{\pi}}\, d\theta$

$$= \frac{2M}{R^{\beta-1}} \frac{\pi}{2\alpha R}\left(1 - e^{-\alpha R}\right)$$

$$= \frac{M\pi}{\alpha R^{\beta}}\left(1 - e^{-\alpha R}\right)$$

§92 Integrals o.t.f. $\int_0^{2\pi} f(\sin t\,;\cos t)\,dt$.

Try the substitution $z = \cos t + i\sin t$.

$\Rightarrow z^{-1} = \cos t - i \sin t$

motivation: $z = e^{it}$ $0 \le t \le 2\pi$

$\Rightarrow$ (check!) $\cos t = \frac{1}{2}(z + z^{-1})$;

$\sin t = \frac{1}{2i}(z - z^{-1})$.

$dz = (-\sin t + i\cos t)\,dt = iz\,dt$.

E.g. find $I = \int_0^{2\pi} \frac{dt}{2+\cos t}$.

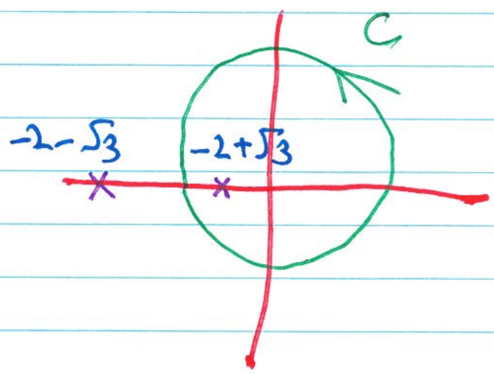
Put $z = e^{it}$: by above, $dt = dz/iz$ &

$\cos t = \frac{1}{2}(z + z^{-1})$

So, for C given by $\{z = e^{it}\ \ 0 \le t \le 2\pi\}$,

we have

$$I = \int_C \frac{dz/iz}{2 + \frac{1}{2}(z + z^{-1})}$$



$$= -i\int_C \frac{dz}{2z + \frac{1}{2}(z^2+1)}$$

$$= -2i\int_C \frac{dz}{z^2 + 4z + 1} \quad (*)$$

To evaluate the integral in $(*)$: note that the integrand is analytic except at the zeros of the denominator, i.e. $-2 \pm \sqrt{3}$. Only $-2+\sqrt{3}$ is inside C.

Res Th^{m} (*) $\Rightarrow$

$$I = (-2i)\, 2\pi i \operatorname*{res}_{z=-2+\sqrt{3}} \overset{f(z)}{\frac{1}{z^2+4z+1}} \quad (**)$$

Near $-2+\sqrt{3}$, $f(z) = \dfrac{p(z)}{q(z)}$, where

$p(z) = 1$, $q(z) = z^2+4z+1$ are analytic,

$p(-2+\sqrt{3}) = 1 \neq 0$.

$q(-2+\sqrt{3}) = 0$

$$q'(-2+\sqrt{3}) = 2z+4\Big|_{-2+\sqrt{3}}$$

$$= 2\sqrt{3} \neq 0.$$

So f has a simple pole at $-2+\sqrt{3}$,

$$\& \operatorname*{res}_{z=-2+\sqrt{3}} f(z) = \frac{p}{q'}\Big|_{-2+\sqrt{3}}$$

$$= 1/2\sqrt{3}.$$

Hence, (**) $\Rightarrow I = (-2i)(2\pi i)\dfrac{1}{2\sqrt{3}} = 2\pi/\sqrt{3}$.

Next Rouché's Th^{m}.