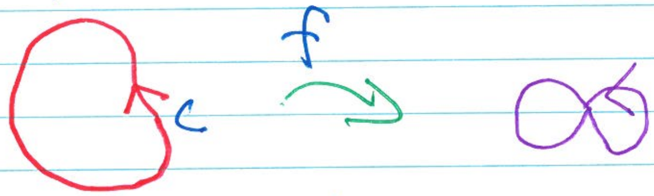
LECTURE 37

Argument Principle & Rouché's Thm §93–94 (8Ed §86–7)

and non-vanishing on C,

f analytic in & on C, a $+$vely oriented, simple closed contour, except possibly for poles inside C. (Note: this means finitely many poles, see Thm 2 §82 (8Ed §75)).



Argument principle: $\Delta_C \arg f = 2\pi(z-p)$

change in the argument of f as we go around C once.

Here: z = # zeros inside C, counting multiplicity

p = # poles inside C, counting total order.

E.g. $\dfrac{1}{(z-\frac{1}{2})^3(z+\frac{i}{6})^5}$ satisfies $p=8$, $z=0$

for C = unit circle, $+$vely oriented;

& $p=5$, $z=0$ for C' = circle of radius $\frac{1}{4}$, centre 0, $+$vely oriented.

Applic: Rouché's Thm: Let f & g be analytic on C & Int C, where C is a simple closed contour, orientation irrelevant.

Suppose $|f(z)| > |g(z)|$ $\forall z \in C$.

Then f & $f+g$ have the same # of zeros, counting multiplicity, inside C.

Note, e.g., $f(z) = (z-i)^2(z+i)^3$ has 5 zeros in $\mathbb{C}$, counting multiplicity.

Example of Rouché: How many zeros of $h(z) = z^7 - 4z^3 + z - 1$ lie inside the unit circle?

Sol^n: Put $f(z) = -4z^3$

$g(z) = z^7 + z - 1$

f & g are entire (polys)

$|f| = 4$ on C since $|z| = 1$ on C

Note: $|g(z)| = |z^7 + z - 1|$

$\leq |z^7| + |z| + |-1|$ Δ-ineq

$= 3 < 4 = |f(z)|$ on C.

So, Rouché $\Rightarrow$ f & $f+g = h$ have the same # of zeros (counting mult.) inside C, namely 3, since f has a 3-fold zero at 0.

"$f(z)$"

Last Q^n: Laurent series for $\frac{1}{\sinh z}$ at 0.

f is not entire, analytic on $\mathbb{C} \setminus \{n\pi i, n \in \mathbb{Z}\}$

So 0 is an isolated sing & f has a Laurent series on $0 < |z| < \pi$.

$$f(z) = \frac{1}{\sinh z} = \frac{1}{z + z^3/3! + z^5/5! + \cdots}$$

$$= \frac{1}{z}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right)$$

$$= \frac{1}{z}\,\frac{1}{1 - \left(-z^2/3! - z^4/5! - \cdots\right)} \quad (*)$$

Geom series sum $\Rightarrow$

for $|z|$ suff small, $|z^2/3! + z^4/5! + \cdots| < 1$

So $(*)$ $= \frac{1}{z}\cdot\left[1 + \left(-z^2/3! - z^4/5! - \cdots\right) + \left(\frac{-z^2}{3!} - z^4/5! - \cdots\right)^2\right]$

$$= \frac{1}{z} - z/6 + \frac{7z^3}{360} + \cdots.$$

$\Rightarrow$ simple pole at 0, with

$$\operatorname*{res}_{z=0} \frac{1}{\sinh z} = 1 \quad (\text{coeff of } z^{-1}).$$

Lecture 18: review up to in-sem.

* line integrals
* contour integrals
* Cauchy-Goursat, Cauchy's integral formula
* Formula for derivatives; analytic $\Rightarrow$ only diff^{ble}.
* Morera's th^m: th^m:
* Liouville – fund th^m of algebra.
* Conformal Maps
* Bdy value problems
* Harmonic f^{n}s
* harmonic conjugates.
* Transformation of b.v.p.s.
* Power series / Taylor series / Laurent series.
* Analytic f^n $\Leftrightarrow$ equal to Taylor series.
* Singularities (isolated / non-isolated).
* Classifying isolated sings:
    * removable
    * pole $\longrightarrow$ ∘ Order.
    * essential.
* Residue, methods for calculating.
* Relationship between multiplicity of zeros & orders of poles.
* Integration on $\mathbb{R}$ using contour integrals, res th^m, passage to limits, properties of PV integrals.

* M-L thᵐ ; Jordan's Lemma
* Arg principle ⇝ Rouché's thᵐ.

not explicitly examinable.

* Verify conditions of thᵐs.
* Check algebra, sanity check.
* remember you are in $\mathbb{C}$.