REVISION SESSION 1

2022 Q7 a) f is a quotient of entire f^{n}s (exp × poly, poly), so is analytic, except for zeros of denominator, i.e., $z = \pm\sqrt{3}i$. These are isolated.

Near $z = \sqrt{3}i$ $\quad f(z) = \dfrac{\phi(z)}{z-\sqrt{3}i}$, where

$$\phi(z) = \frac{z e^{2iz}}{z+\sqrt{3}i}$$ is analytic & $\neq 0$

(indeed $\phi(\sqrt{3}i) = \dfrac{\sqrt{3}i\, e^{-2\sqrt{3}}}{2\sqrt{3}i} = e^{-2\sqrt{3}}/2$).

Hence f has a simple pole at $\sqrt{3}i$.

Similarly, near $z = -\sqrt{3}i$, $f(z) = \dfrac{\psi(z)}{z+\sqrt{3}i}$, where

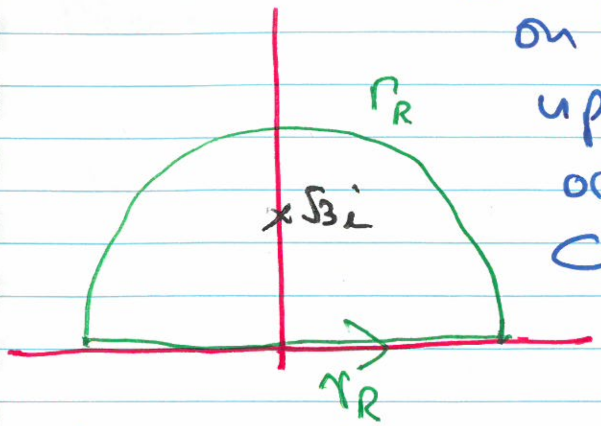
$$\psi(z) = \frac{z e^{2iz}}{z-\sqrt{3}i}$$ is analytic & $\neq 0$.

Hence, f has a simple pole at $-\sqrt{3}i$.

b) Since f has a simple pole at $\sqrt{3}i$,

$$\operatorname*{res}_{z=\sqrt{3}i} f = \phi(\sqrt{3}i) = e^{-2\sqrt{3}}/2.$$

c) Consider $R>\sqrt{3}$ & a contour $C = C_R + \gamma_R$ as shown: γ_R is the line segment from $-R$ to R on the real axis; C_R is the upper semi-circle, radius R, +vely oriented. f is analytic in & on C, except for the isolated sing. at $\sqrt{3}i$.



Hence, residue thm $\Rightarrow$

$$\int_C f = 2\pi i \operatorname*{res}_{z=\sqrt{3}i} f \overset{(b)}{=} 2\pi i \cdot \frac{1}{2} e^{-2\sqrt{3}} = \pi i e^{-2\sqrt{3}}. \quad \circledast$$

$$\text{LHS of } \circledast = \int_C f = \underbrace{\int_{C_R} f}_{= I_R} + \underbrace{\int_{-R}^{R} f}_{= \mathcal{I}_R}$$

$$|I_R| = \left| \int_{C_R} \frac{z e^{2iz}}{z^2+3} dz \right|.$$

On C_R, $|e^{2iz}| = |e^{2i(x+iy)}| = |e^{-2y}||e^{2ix}| = e^{-2y} \leq 1$, but M-L won't work to estimate $|I_R|$, because $|f| \lesssim 1/R$, $l \lesssim R$.

Try for Jordan's Lemma.

Note $f(z) = g(z)e^{2iz} = g(z)e^{iaz}$, where $a=2>0$.

Here $g(z) = \frac{z}{z^2+3}$ is analytic on $\{z : \mathrm{Im}(z) \geq 0\} \cap \{z : |z| > \sqrt{3}\}$

Further, on C_R, $|g(z)| = \left|\frac{z}{z^2+3}\right| \leq \frac{R}{R^2-3}$

$$= \frac{1}{R - 3/R} = M_R, \text{ with } M_R \to 0 \text{ as } R \to \infty.$$

Hence by Jordan's Lemma, $\lim_{R\to\infty} |I_R| = 0$

Hence also $\lim_{R\to\infty} \mathrm{Im}(I_R) = 0$

Note also $\mathrm{Im}(I_R) = \int_{-R}^{R} \mathrm{Im}(f(x))\,dx = \int_{-R}^{R} \frac{x\sin 2x}{x^2+3}\,dx$

So, taking imaginary parts in $(*)$ & letting $R\to\infty$, we have

$$\pi e^{-2\sqrt{3}} = \int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+3}\,dx = 2\int_0^{\infty} \frac{x\sin 2x}{x^2+3}\,dx,$$

since the integrand is even.

Hence $\int_0^{\infty} \frac{x\sin 2x}{x^2+3}\,dx = \frac{\pi}{2} e^{-2\sqrt{3}}$ as req^d.