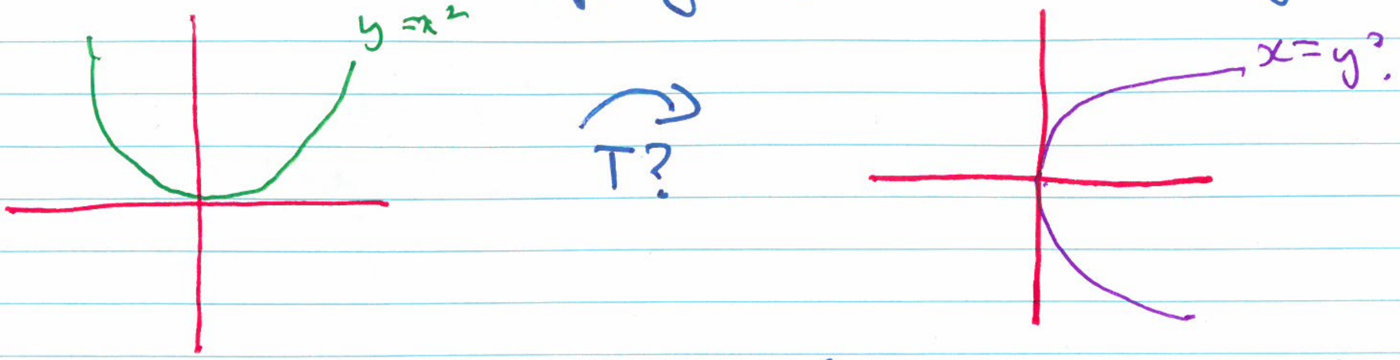
REVISION SESSION 2

2024 9e)

Möb transform mapping $\{y = x^2\}$ to $\{x = y^2\}$.



T corresponds to rotation (about the origin) by $-\pi/2$, i.e., multiplying by $e^{-i\pi/2} = -i$.

$T(z) = -iz$, which is a Möb. transf. $(a = -i,\ b = c = 0,\ d = 1)$.

2020 q 5a).

$$f(z) = \frac{(1-\cos z)^3}{z^9}$$

(i) f is a quotient of entire f^n's, so is analytic on $\mathbb{C} \setminus \{0\}$, since 0 is the only zero of the denominator. ⇒ 0 is an isolated singularity.

(ii) $$f(z) = \frac{1}{z^9}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\right]^3$$
$$= \frac{1}{z^9}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)^3$$
$$= \frac{1}{z^9}\left(\frac{z^6}{2!\,2!\,2!} - \frac{3z^8}{2!\,2!\,4!} + \cdots\right)$$
$$= \frac{1}{8z^3} - \frac{1}{32z} + \ldots$$

(iii) Singularity at 0 is a pole of order 3 (finitely many -ve powers of z : 3 is greatest -ve power.

(iv) residue $= -\frac{1}{32} =$ coeff of z^{-1}.

2024 q6.

For $f(z) = C_1 \sin(e^{-i\pi/4} z \cdot \pi)$, we have zeros for $e^{-i\pi/4} z\pi = n\pi$, $n \in \mathbb{Z}$

i.e, $z = ne^{i\pi/4}$ so adjacent zeros are separated by 1 & are on the line $y=x$, & $f(0)=0$.
Thus f satisfies (i) & (ii)

For $z = \frac{e^{i\pi/4}}{2}$, we have $f(z) = C_1$

So for $C_1 = 2$:

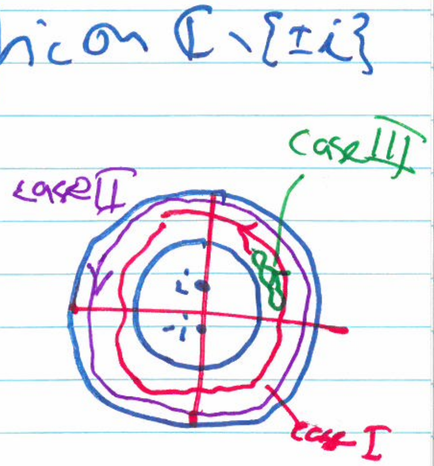
$$f(z) = 2\sin(e^{-i\pi/4} z\pi) \quad \text{suffices.}$$

b) No Eg, Multiply by any f^n only vanishing at (some of the) zeros of f, & $=1$ at $\frac{e^{i\pi/4}}{2}$

e.g. $g(z) = \frac{2z}{e^{i\pi/4}}$

Then $f \cdot g$ also satisfies the cond^ns (i)-(iii).

2022 q4 $f(z) = \frac{1}{z^2+1}$ is analytic on $\mathbb{C} \setminus \{\pm i\}$
In particular, it is analytic on C.
Since $C \subset D$, one of 3 cases occur.



(I) $i, -i \in \operatorname{Int} C$, C +vely oriented

(II) " " " " C -vely oriented.

(III) $i \notin \operatorname{Int} C$, $-i \notin \operatorname{Int} C$

Claim: in all cases, $\int_C f = 0$.

In case (III), this follows from Cauchy-Goursat.

In case I: note f has simple poles at $\pm i$,
since $z^2+1 = (z-i)(z+i)$.

f can be written as $\frac{p}{q}$, with $p(z) \equiv 1 \neq 0$.
& $q(z) = z^2+1 \Rightarrow q'(z) = 2z$, which doesn't vanish at $-i$ or i.

So
$$\operatorname*{res}_{z=i} f = \frac{p(i)}{q'(i)} = \frac{1}{2i}$$
$$\operatorname*{res}_{z=-i} f = \frac{p(-i)}{q'(-i)} = \frac{1}{-2i}$$

So res thm $\Rightarrow \int_C f = 2\pi i\left(\frac{1}{2i} - \frac{1}{2i}\right) = 0$.

In case II same, but $\int_{-C} f = 0 \Rightarrow \int_C f = 0$.

Could also use partial fractions, or $\frac{\phi(z)}{z-i}$ etc.