REVISION SESSION I

§39 Q16. (see Lec 10)

Ans: §37 (14) $\Rightarrow \cos z = \cos x \cosh y - i \sin x \sinh y$

$\cos z = 2 \Rightarrow \begin{cases} \cos x \cosh y = 2 \text{ ① } \& \\ \sin x \sinh y = 0 \text{ ②} \end{cases}$

② $\Rightarrow y = 0$ Ⓐ or $x = n\pi$. Ⓑ

Ⓐ $\Rightarrow$ ① becomes $\cos x = 2$ no solⁿ.

Ⓑ $\Rightarrow x = n\pi \Rightarrow \cos x = (-1)^n$:

n odd $\Rightarrow \cosh y = -2$, no solⁿ.

n even $\Rightarrow \cosh y = 2$.

So solⁿ. is $z = 2n\pi + i\cosh^{-1}2, \ n \in \mathbb{Z}$.

In general: $w = \cosh^{-1} z \Leftrightarrow z = \cosh w = \frac{e^w + e^{-w}}{2}$.

$\Rightarrow 2z = e^w + e^{-w} \overset{\cdot e^w}{\Rightarrow} 2ze^w = e^{2w} + 1$.

$\Rightarrow e^{2w} - 2ze^w + 1 = 0$.

$\Rightarrow e^w = \frac{2z + (4z^2 - 4)^{1/2}}{2}$ $\quad \forall z \in \mathbb{C}$.

$= z + (z^2 - 1)^{1/2}$

So $w = \cosh^{-1} z = \log(e^w) = \log(z + (z^2-1)^{1/2})$

for $z = 2$: $w = \cosh^{-1} 2 = \log(2 + (2^2 - 1)^{1/2})$

$= \log(2 + 3^{1/2})$

$= \{\log(2+\sqrt{3}), \log(2-\sqrt{3})\}$.

$= \{\ln|2+\sqrt{3}| + 2n\pi i, \ \ln|2-\sqrt{3}| + 2n\pi i, \ n \in \mathbb{Z}\}$

$= \{\ln(2+\sqrt{3}) + 2n\pi i, \ \ln(2-\sqrt{3}) + 2n\pi i; \ n \in \mathbb{Z}\}$. ⊛

Note: $\ln(2-\sqrt{3}) = \ln\left[(2-\sqrt{3}) \cdot \frac{(2+\sqrt{3})}{(2+\sqrt{3})}\right] = \ln\left(\frac{1}{2+\sqrt{3}}\right) = -\ln(2+\sqrt{3})$

⊛ $\Rightarrow \cosh^{-1} 2 = \pm \ln(2+\sqrt{3}) + 2n\pi i$.

2023 Q6: a) $PV\int_{-\infty}^{\infty} f(x)\,dx = \lim_{M\to\infty} \int_{-M}^{M} f(x)\,dx$. (*)

b) Put $f(x) = \frac{x^7}{(x^4+1)^2}$. Note f is defined on all $\mathbb{R}$ (rational f^n, denominator never vanishes), & is odd.

Hence $\int_{-M}^{0} f = -\int_{0}^{M} f \ \forall M \Rightarrow \int_{-M}^{M} f = 0 \ \forall M > 0$.

So from (*) $PV\int_{-\infty}^{\infty} f = \lim_{M\to\infty} 0 = 0$.

c) $f \sim 1/x$ as $x \to \pm\infty$, so $\lim_{M\to\infty}\int_0^M f$ & $\lim_{N\to-\infty}\int_N^0 f$ are undefined, hence $\int_{-\infty}^{\infty} f$ is undefined.

2025 Q6 b):

(i) $g(z) = \frac{\cos z}{z^2 \sin z}$ (*), so g is analytic except for zeros of denom., i.e. $z=0$ & $z = n\pi$, $n \in \mathbb{Z}$.

Hence, 0 is an isolated sing.

(ii) $g(z) = \frac{(1 - z^2/2! + \cdots)}{z^2(z - z^3/3! + \cdots)}$

$= \frac{1}{z^3}\left(\frac{1 - z^2/2! + \cdots}{1 - z^2/3! + \cdots}\right)$

$= \frac{1}{z^3}(1 - z^2/2! + \cdots)\cdot\frac{1}{1-(z^2/3! - z^4/5! + \cdots)}$

for $|z|$ small

$= \frac{1}{z^3}(1 - z^2/2! + \cdots)(1 + (z^2/3! - \cdots) + (z^2/3! + \cdots)^2 + \cdots)$

(**)

$= \frac{1}{z^3} - \frac{1}{3z} + \cdots$

(iii) from (ii) z has a pole of order 3 at 0 (greatest negative power).

(iv) $\underset{z=0}{\text{res}}\, g(z) =$ coeff of z^{-1} in ~~...~~ $= -\frac{1}{3}$.

2021 Q6 a). f has the form $\frac{p}{q}$, where p & q are entire (rational f^n). Singularities are zeros of denom. Therefore

$$q(z) = (z-ai)(z+ai)(z-bi)(z+bi) \text{ \& } p$$

is non vanishing on $\mathbb{C}$. Hence f has simple poles at each of the zeros of q, i.e., $\pm ai$, $\pm bi$.