REVISION SESSION 2.

2021 Q2 $f(z) = 2\bar{z} - |z|^2 \bar{z}$

a) Note $f(z) = 2\bar{z} - z\bar{z}^2$

C/R (Wirtinger) $\Rightarrow \partial_{\bar{z}} f = 0$.

$\Rightarrow 2 - 2\cancel{z\bar{z}} = 0$.

$2(1-|z|^2) = 0$.

So C/R hold precisely on $\{z : |z| = 1\}$. (*)

Can also rewrite & use standard C/R.

b) $f(\overset{x+iy}{\overset{"}{z}}) = 2(x-iy) - (x^2+y^2)(x-iy)$

is a product of polynomials in x & y, so u & v have cts partials of all orders.

Combined with (*) $\Rightarrow f$ is diff^{ble} on $\{z : |z| = 1\}$.

c) There is no nbhd on which f is diff^{ble}.

d) analytic $\Rightarrow$ diff^{ble}.

$\not\Leftarrow$

$f(z) = \frac{1}{z^2+z} = \frac{1}{z(z+1)}$ is analytic on $\mathbb{C} \setminus \{0,-1\}$. In particular, it is analytic on C.

$\int_C f(z)\,dz = Ⓐ$

1st group: C +vely oriented.

Case I $\quad 0, -1 \in \text{Int } C$.

Case II $\quad 0, -1 \in \text{Ext } C$.

Case III $\quad 0 \in \text{Int } C, -1 \in \text{Ext } C$.

Case IV $\quad -1 \in \text{Int } C, 0 \in \text{Ext } C$.

Case II : Ⓐ $= 0$ by Cauchy-Goursat.

Want to calculate $\operatorname*{res}_{z=0,-1} f(z)$.

$$f(z) = \frac{p(z)}{q(z)} \quad \text{where } p(z) \equiv 1$$

& $q(z) = z^2+z$, so $q'(z) = 2z+1 \neq 0$ at either 0 or -1.

$\Rightarrow$ Simple poles at 0 & -1, and

$$\operatorname*{res}_{z=0} f(z) = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

& $$\operatorname*{res}_{z=-1} f(z) = \frac{p(-1)}{q'(-1)} = \frac{1}{-1} = -1.$$

So in case I, (A) $= (1 + -1) \cdot 2\pi i$ by Cauchy
$= 0$

Similarly in case III (A) $= 2\pi i$,
in case IV (A) $= -2\pi i$.

2nd group, C -vely oriented:
case I, case II (A) $= 0$, case III (A) $= -2\pi i$,
case IV (A) $= 2\pi i$.

So, possible values are $\{-2\pi i, 0, 2\pi i\}$.

2021 q5 Put $f = u + iv$, so $\bar{f} = w + ih$,
where $w = u$ & $h = -v$. (1)
Since $\bar{f}$ is analytic, it satisfies C/R.
hence $w_x = h_y$ & $w_y = -h_x$ (2).

(1) & (2) $u_x = -v_y$ & $u_y = v_x$ (3).
Since f is analytic, it satisfies C/R,
hence $u_x = v_y$ & $u_y = -v_x$ (4)

(3) & (4) $\Rightarrow$ $v_y = -v_y \Rightarrow v_y = 0$
& $v_x = -v_x \Rightarrow v_x = 0$.

So v is constant, so via (4), $u_x = u_y = 0 \Rightarrow u$ is constant $\Rightarrow f$ is constant.

b). Suppose $|g(z)| = \gamma$ ① on Ω, $\gamma \geq 0$.
If $\gamma = 0 \Rightarrow g = 0 \Rightarrow \bar{g} = 0$ which is constant.
If $\gamma > 0$, ① $\Rightarrow g(z) \cdot \bar{g}(z) = \gamma^2 \; \forall z \in \Omega$

$\Rightarrow \bar{g}(z) = \dfrac{\gamma^2}{g(z)}$ note g doesn't vanish on Ω, since $\gamma \neq 0$.

$\Rightarrow \bar{g}$ is analytic on Ω, since g is.
But by a), since g & $\bar{g}$ are analytic on Ω, g is constant.

6 a) $f(z) = C_1 \sin(e^{-i\pi/4} z\pi)$ has zeros for $e^{-i\pi/4} z = n$, $n \in \mathbb{Z}$, i.e.,

$z = n e^{i\pi/4}$, so adjacent zeros are separated by unit distance.

This satisfies (i) & (ii)

For $z = e^{i\pi/4}/2$, we have $f(z) = C_1$.

So, for $C_1 = 2$, $f(z) = 2\sin(e^{-i\pi/4} z\pi)$ suffices.

b) No. Multiply f by any f^n that only vanishes at zeros of f, $\Delta = 1$ at $e^{i\pi/4}/2$:
e.g. $g(z) = 2z/e^{i\pi/4}$; $f \cdot g$ also satisfies (i)–(iii).