# MATH2001, MATH2901 & MATH7000

Lecture Workbook

# Contents

1	Solu	tions of first order ODEs	12
	1.1	Example: $\frac{dy}{dx} = x, y(0) = 1$ has a unique solution	13
	1.2	Example: $\frac{dy}{dx} = 3xy^{1/3}$ , $y(0) = 0$ has more than one solution	13
	1.3	Example: $\frac{dy}{dx} = \frac{x-y}{x}$ , $y(0) = 1$ has no solution	14
	1.4	Existence and uniqueness criteria	14
	1.5	Example: $\frac{dy}{dx} = x, y(0) = 1$	15
	1.6	Example: $\frac{dy}{dx} = 3xy^{1/3}, y(0) = 0$	15
	1.7	Example: $\frac{dy}{dx} = \frac{x-y}{x}$ , $y(0) = 1$	15
<b>2</b>	Me	thod of successive approximations	17
	2.1	A modified theorem	17
	2.2	2.1.1 Example: $y' = 2(x - 1)(y - 1), y(1) = 2$	17
	2.2	Equivalence with integral equation	18
	2.3	Method of successive approximations $\dots \dots \dots$	18
	$2.4 \\ 2.5$	Example: $y = 2x(y+1), y(0) = 0$ Discussion	$\frac{19}{20}$
3	Exa	ct first order ODEs	25
	3.1	Definition	25
	3.2	Test for exactness	26
	3.3	Example: $2x + e^y + xe^y y' = 0$	26
	3.4	Almost exact ODEs and integrating factors	27
	3.5	Example: $(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$	28
4	Hyj	perbolic functions	30
	4.1	Properties of hyperbolic functions	30
	4.2	Inverse hyperbolic functions	34
		4.2.1 Show that $\frac{d}{dx} (\operatorname{arsinh}(x)) = \frac{1}{\sqrt{1+x^2}} \dots \dots \dots \dots$	36
		4.2.2 Evaluate the integrals $\int \frac{dx}{\sqrt{1+x^2}}$ and $\int \frac{dx}{\sqrt{x^2-1}}$	37
		4.2.3 Show that $\frac{d}{dx}(\operatorname{artanh}(x)) = \frac{1}{1-x^2},  x  < 1$	38
		4.2.4 Show that $\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$	39
	4.3	Further reading: The catenary problem	40
<b>5</b>	Lin	ear second order nonhomogeneous ODEs, the Wronskian	43
	5.1	Existence and uniqueness theorem	43
	5.2	The superposition principle	44

	5.3	Linear independence
		5.3.1 Example: linear independence of $g_1(x) = 1$ , $g_2(x) = x$ , $g_3(x) =$
		$x^2$ on the interval $-\infty < x < \infty$
	5.4	General solution (homogeneous case), Wronskian of solutions
	5.5	Abel's Theorem
	5.6	Nonhomogenous linear ODEs 48
		5.6.1 Example: $y'' - 2y' + y = e^x + x$
	5.7	Further reading: The method of undetermined coefficients 49
6	Var	ation of parameters 51
	6.1	Derive the formulae for $u(x)$ and $v(x)$ in the variation of parameters 52
	6.2	Example: $y'' - 4y' + 5y = 2e^{2x} / \sin x$
	6.3	Repeat the previous example using reduction of order
	6.4	Summary of ODE techniques and types of equations you should know . 55
7	Vec	tor spaces 57
	7.1	Definition
	7.2	Example: $\mathbb{F}^n$ – set of <i>n</i> -tuples
	7.3	Example: $M_{m,n}(\mathbb{F})$ – set of $m \times n$ matrices
	7.4	Example: $C[a, b]$ – set of continuous real-valued functions on $[a, b]$ 59
	7.5	Example: $P_n(\mathbb{F})$ – set of polynomials of degree at most $n$
	7.6	Example: Set of solutions to a homogeneous linear ODE
	7.7	Familiar concepts in linear algebra 60
	7.8	Basis
	7.9	Decomposition theorem
	7.10	Transition matrix
8	Rea	l inner product spaces 65
	8.1	Dot product $\ldots$
	8.2	Inner product
	8.3	Example: Weighted dot product
	8.4	Example: Inner product generated by a matrix
	8.5	Example: Inner product on $M_{n,n}(\mathbb{R})$
	8.6	Example: Standard inner product on $P_n(\mathbb{R})$
	8.7	Example: Evaluation inner product on $P_n(\mathbb{R})$
	8.8	Example: Inner product on $C[a, b]$
9	Solv	ring linear equations 72
	9.1	Gaussian Elimination
		9.1.1 Example
		9.1.2 Definition (row echelon form)
		9.1.3 Elementary row operations
	9.2	Possible solutions for $A\boldsymbol{x} = \boldsymbol{b}$
		9.2.1 Examples

10.1 Norm       79         10.2 Pythagorean theorem       80         10.3 Cauchy-Schwarz inequality       80         10.4 Triangle inequality       81         10.5 Example: $C[-1, 1]$ 81         10.6 Example: $P_2(\mathbb{R})$ 81         10.7 Angle between two vectors       82         10.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product       83         11 Gram-Schmidt process       85         11.2 Orthogonal set       85         11.2 Orthogonal projection       88         11.5 Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.6 Construction of orthonormal basis for $P_1(\mathbb{R})$ 91         12 Least squares problem - minimising distance to a subspace       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 Col( $A)^{\perp} = N(A^{T})$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       102         13.2 Fourier coefficients       103         14 Understanding determinants       106         14.1 The effect of a linear transformation on a grid       106 </th <th>10 Orthogonality</th> <th>79</th>	10 Orthogonality	79
10.2Pythagorean theorem8010.3Cauchy-Schwarz inequality8010.4Triangle inequality8110.5Example: $P_2(\mathbb{R})$ 8110.6Example: $P_2(\mathbb{R})$ 8110.7Angle between two vectors8210.8Orthogonal complement8310.8.1For $A \in M_{m,n}(\mathbb{R}), \operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product8311Gram-Schmidt process8511.1Orthogonal set8511.2Orthonormal basis8611.3Decomposition theorem8711.4Orthogonal projection8811.5Example: Orthonormal basis9011.7Example: Orthonormal basis9011.7Example: Orthonormal basis for $P_1(\mathbb{R})$ 9112Least squares approximation9312.1Least squares problem - minimising distance to a subspace9312.2Inconsistent linear systems9412.3Col( $A)^{\perp} = N(A^T)$ 9612.4Solving for $\hat{x}$ 9712.5Fitting a curve to data9813Least squares function approximation10113.1Example: $\sin(x)$ 10213.2Fourier coefficients10314Understanding determinants10514.1The effect of a linear transformation on a grid10614.2The determinant and orientation10914.4Further examples10115.2.1Exampl	10.1 Norm	7
10.3 Cauchy-Schwarz inequality       80         10.4 Triangle inequality       81         10.5 Example: $C[-1, 1]$ 81         10.6 Example: $P_2(\mathbb{R})$ 81         10.7 Angle between two vectors       82         10.8 Orthogonal complement       83         10.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product       83         11 Gram-Schmidt process       85         11.1 Orthogonal set       85         11.2 Orthonormal basis       86         11.3 Decomposition theorem       87         11.4 Orthogonal projection       88         11.5 Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.6 Construction of orthonormal basis for $P_1(\mathbb{R})$ 91         12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Florensistent linear systems       94         12.3 Col( $A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $\sin(x)$ 102	10.2 Pythagorean theorem	80
10.4 Triangle inequality8110.5 Example: $C[-1, 1]$ 8110.6 Example: $P_2(\mathbb{R})$ 8110.7 Angle between two vectors8210.8 Orthogonal complement8310.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product8311 Gram-Schmidt process8511.1 Orthogonal set8511.2 Orthonormal basis8611.3 Decomposition theorem8711.4 Orthogonal projection8811.5 Example: Orthogonal projection in $\mathbb{R}^3$ 8911.6 Construction of orthonormal basis9011.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 9112 Least squares approximation9312.1 Least squares problem - minimising distance to a subspace9312.2 Inconsistent linear systems9412.3 Col( $A ^{\perp} = N(A^T)$ 9612.4 Solving for $\hat{\mathbf{x}}$ 9712.5 Fitting a curve to data9813 Least squares function approximation10113.1 Example: $\sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10814.3 Sign of the determinant and orientation10915 Eigenvalues and eigenvectors11115.1 Non-singular matrices11115.2.1 Example11315.2.1 Example11315.2.1 Example114	10.3 Cauchy-Schwarz inequality	80
10.5 Example: $C[-1, 1]$ 8110.6 Example: $P_2(\mathbb{R})$ 8110.7 Angle between two vectors8210.8 Orthogonal complement8310.8.1 For $A \in M_{m,n}(\mathbb{R})$ , Row $(A)^{\perp} = N(A)$ with respect to the Euclidean inner product8311 Gram-Schmidt process8511.1 Orthogonal set8511.2 Orthonormal basis8611.3 Decomposition theorem8711.4 Orthogonal projection8811.5 Example: Orthonormal basis9011.7 Example: Orthonormal basis9011.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 9112 Least squares approximation9312.1 Least squares problem - minimising distance to a subspace9312.2 Inconsistent linear systems9412.3 Col( $A$ ) <sup>⊥</sup> = $N(A^T)$ 9612.4 Solving for $\hat{\mathbf{x}}$ 9712.5 Fitting a curve to data9813 Least squares function approximation10113.1 Example: $\sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10914.4 Further examples11115.1 Non-singular matrices11115.2 Eigenvalues and eigenvectors11115.2 I. Example11315.3 Simple properties114	10.4 Triangle inequality	8
10.6 Example: $P_2(\mathbb{R})$ 8110.7 Angle between two vectors8210.8 Orthogonal complement8310.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product8311 Gram-Schmidt process8511.1 Orthogonal set8511.2 Orthonormal basis8611.3 Decomposition theorem8711.4 Orthogonal projection8811.5 Example: Orthogonal projection in $\mathbb{R}^3$ 8911.6 Construction of orthonormal basis9011.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 9112 Least squares approximation9312.1 Least squares problem - minimising distance to a subspace9312.2 Inconsistent linear systems9412.3 Col( $A)^{\perp} = N(A^T)$ 9612.4 Solving for $\hat{\mathbf{x}}$ 9712.5 Fitting a curve to data9813 Least squares function approximation10113.1 Example: $\sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10914.4 Further examples10915 Eigenvalues and eigenvectors11115.2 Eigenvalues and eigenvectors11215.3 Simple properties11414.1 The effect of a linear transformation on a grid to the determinant and orientation10914.4 Further examples11315.2 Eigenvalues and eigenvectors11415.4 Fibrometere accurate and bias to the total state	10.5 Example: $C[-1,1]$	8
10.7 Angle between two vectors8210.8 Orthogonal complement8310.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product8311 Gram-Schmidt process8511.1 Orthogonal set8511.2 Orthonormal basis8611.3 Decomposition theorem8711.4 Orthogonal projection8811.5 Example: Orthogonal projection in $\mathbb{R}^3$ 8911.6 Construction of orthonormal basis9011.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 9112 Least squares approximation9312.1 Least squares problem - minimising distance to a subspace9312.2 Inconsistent linear systems9412.3 Col $(A)^{\perp} = N(A^T)$ 9613 Least squares function approximation10113.1 Example: sin $(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant and orientation10914.4 Further examples10915.1 Non-singular matrices11115.2 Eigenvalues and eigenvectors11215.3 Simple properties11315.3 Simple properties114	10.6 Example: $P_2(\mathbb{R})$	8
10.8 Orthogonal complement8310.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product8311 Gram-Schmidt process8511.1 Orthogonal set8511.2 Orthonormal basis8611.3 Decomposition theorem8711.4 Orthogonal projection8811.5 Example: Orthogonal projection in $\mathbb{R}^3$ 8911.6 Construction of orthonormal basis9011.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 9112 Least squares approximation9312.1 Least squares problem - minimising distance to a subspace9312.2 Inconsistent linear systems9412.3 Col( $A)^{\perp} = N(A^T)$ 9612.4 Solving for $\hat{\mathbf{x}}$ 9712.5 Fitting a curve to data9813 Least squares function approximation10113.1 Example: $\sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10814.3 Sign of the determinant and orientation10914.4 Further examples10915.2 Eigenvalues and eigenvectors11115.1 Non-singular matrices11115.2.1 Example11315.3 Simple properties114	10.7 Angle between two vectors	8
10.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product       83         11 Gram-Schmidt process       85         11.1 Orthogonal set       85         11.2 Orthonormal basis       86         11.3 Decomposition theorem       87         11.4 Orthogonal projection       88         11.5 Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.6 Construction of orthonormal basis       90         11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 91         12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 Col( $A$ ) <sup><math>\perp</math></sup> = $N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $\sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       109         14.4 Further examples       109         15.1 Non-singular matrices       111         15	10.8 Orthogonal complement	
clidean inner product8311 Gram-Schmidt process8511.1 Orthogonal set8511.2 Orthonormal basis8611.3 Decomposition theorem8711.4 Orthogonal projection8811.5 Example: Orthogonal projection in $\mathbb{R}^3$ 8911.6 Construction of orthonormal basis9011.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 9112 Least squares approximation9312.1 Least squares problem - minimising distance to a subspace9312.2 Inconsistent linear systems9412.3 $Col(A)^{\perp} = N(A^T)$ 9612.4 Solving for $\hat{\mathbf{x}}$ 9712.5 Fitting a curve to data9813 Least squares function approximation10113.1 Example: $sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10814.3 Sign of the determinant and orientation10914.4 Further examples10915.2 Eigenvalues and eigenvectors11115.2 Eigenvalues and eigenvectors11215.3 Simple properties11315.3 Simple properties114	10.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the B	Lu-
11 Gram-Schmidt process       85         11.1 Orthogonal set       85         11.2 Orthonormal basis       86         11.3 Decomposition theorem       87         11.4 Orthogonal projection       88         11.5 Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.6 Construction of orthonormal basis       90         11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 91         12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 $Col(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15.2 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors	clidean inner product	8
11. Orthogonal set       85         11. Orthogonal set       85         11. Orthogonal basis       86         11. Orthogonal projection theorem       87         11. Orthogonal projection       88         11. Casample: Orthonormal basis       90         11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 91         12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 Col( $A^{1} = N(A^{T})$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: sin( $x$ )       102         13.2 Fourier coefficients       103         14.1 The effect of a linear transformation on a grid <td>11 Gram-Schmidt process</td> <td>8</td>	11 Gram-Schmidt process	8
11.1 Orthogonal determinant       86         11.2 Orthonormal basis       86         11.3 Decomposition theorem       87         11.4 Orthogonal projection       88         11.5 Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.6 Construction of orthonormal basis       90         11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 91 <b>12 Least squares approximation</b> 93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 Col( $A^{1\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98 <b>13 Least squares function approximation</b> 101         13.1 Example: $\sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       109         14.4 Further examples       109         14.4 Further examples       109         15.2 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors       112         15.3 Simple properties       114	11 1 Orthogonal set	8
11.2 Ortinionial basis       87         11.3 Decomposition theorem       87         11.4 Orthogonal projection       88         11.5 Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.6 Construction of orthonormal basis       90         11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 91 <b>12 Least squares approximation</b> 93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 $Col(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98 <b>13 Least squares function approximation</b> 101         13.1 Example: $sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15.2 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors       112         15.3 Simple properties       113         15.3 Simple properties	11.1 Orthogonal basis	
11.3       Decomposition theorem       88         11.4       Orthogonal projection       88         11.5       Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.6       Construction of orthonormal basis       90         11.7       Example: Orthonormal basis for $P_1(\mathbb{R})$ 91         12       Least squares approximation       93         12.1       Least squares problem - minimising distance to a subspace       93         12.2       Inconsistent linear systems       94         12.3       Col( $A$ ) <sup>⊥</sup> = $N(A^T)$ 96         12.4       Solving for $\hat{\mathbf{x}}$ 97         12.5       Fitting a curve to data       98         13       Least squares function approximation       101         13.1       Example: $\sin(x)$ 102         13.2       Fourier coefficients       103         14       Understanding determinants       105         14.1       The determinant       108         14.3       Sign of the determinant and orientation       109         14.4       Further examples       109         14.4       Further examples       109         15.2       Eigenvalues and eigenvectors       111         15.2	11.2 Orthonorman basis	
11.4 Orthogonal projection       11.5 Example: Orthogonal projection in $\mathbb{R}^3$ 89         11.5 Example: Orthonormal basis       90         11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 91         12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 $Col(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         14.4 Further examples       109         15.2 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors       112         15.3 Simple properties       113         15.3 Simple properties       114	11.4 Orthogonal projection	
11.5 Example: Orthogonal projection if $\mathbb{R}$ 90         11.6 Construction of orthonormal basis	11.4 Orthogonal projection in $\mathbb{D}^3$	00
11.0 Construction of orthonormal basis       90         11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$ 91         12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 $Col(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114	11.5 Example. Orthogonal projection in $\mathbb{R}$	0
11.1 Example: Orthonomial basis for $T_1(\mathbb{K})$ 93         12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 $Col(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15.5 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors       111         15.2 Lizenvectors       112         15.3 Simple properties       113         15.3 Simple properties       114	11.0 Construction of orthonormal basis	
12 Least squares approximation       93         12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 $Col(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       108         14.3 Sign of the determinant       109         14.4 Further examples       109         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       111         15.3 Simple properties       113         15.3 Simple properties       114	11.7 Example: Orthonormal basis for $F_1(\mathbb{R})$	9
12.1 Least squares problem - minimising distance to a subspace       93         12.2 Inconsistent linear systems       94         12.3 $Col(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114	12 Least squares approximation	93
12.2 Inconsistent linear systems       94         12.3 $\operatorname{Col}(A)^{\perp} = N(A^T)$ 96         12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $\sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114	12.1 Least squares problem - minimising distance to a subspace $\ldots$ .	93
12.3 $\operatorname{Col}(A)^{\perp} = N(A^T)$ 9612.4 Solving for $\hat{\mathbf{x}}$ 9712.5 Fitting a curve to data9813 Least squares function approximation10113.1 Example: $\sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10814.3 Sign of the determinant and orientation10914.4 Further examples10915 Eigenvalues and eigenvectors11115.2 Eigenvalues and eigenvectors11215.2.1 Example11315.3 Simple properties114	12.2 Inconsistent linear systems	
12.4 Solving for $\hat{\mathbf{x}}$ 97         12.5 Fitting a curve to data       98         13 Least squares function approximation       101         13.1 Example: $\sin(x)$ 102         13.2 Fourier coefficients       103         14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114	12.3 $\operatorname{Col}(A)^{\perp} = N(A^T) \dots \dots$	9
12.5 Fitting a curve to data9813 Least squares function approximation10113.1 Example: $sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10814.3 Sign of the determinant and orientation10914.4 Further examples10915 Eigenvalues and eigenvectors11115.2 Eigenvalues and eigenvectors11215.2.1 Example11315.3 Simple properties114	12.4 Solving for $\hat{\mathbf{x}}$	
13 Least squares function approximation10113.1 Example: $sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10814.3 Sign of the determinant and orientation10914.4 Further examples10915 Eigenvalues and eigenvectors11115.2 Eigenvalues and eigenvectors11215.2.1 Example11315.3 Simple properties114	12.5 Fitting a curve to data	98
10 Letter squares removed upproximation10113.1 Example: $sin(x)$ 10213.2 Fourier coefficients10314 Understanding determinants10514.1 The effect of a linear transformation on a grid10614.2 The determinant10814.3 Sign of the determinant and orientation10914.4 Further examples10915 Eigenvalues and eigenvectors11115.2 Eigenvalues and eigenvectors11215.2.1 Example11315.3 Simple properties114	13 Least squares function approximation	10 <sup>.</sup>
13.2 Fourier coefficients       103         14 Understanding determinants       103         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       106         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors       111         15.2 Eigenvalues and eigenvectors       112         15.3 Simple properties       113         15.4 Firzenzectors       114	13.1 Example: $\sin(x)$	10
14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15 Eigenvalues and eigenvectors       109         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114	13.2 Fourier coefficients	10
14 Understanding determinants       105         14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15 Eigenvalues and eigenvectors       111         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.3 Simple properties       113         15.4 First scatter corresponding to distingt signarulues are linearly independent 115		10
14.1 The effect of a linear transformation on a grid       106         14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15 Eigenvalues and eigenvectors       109         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       111         15.2.1 Example       113         15.3 Simple properties       114         15.4 Figenvectors       114	14 Understanding determinants	10!
14.2 The determinant       108         14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15 Eigenvalues and eigenvectors       109         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.3 Simple properties       113         15.4 Firenewaters compared by the distinct eigenvalues are linearly independent 115	14.1 The effect of a linear transformation on a grid	10
14.3 Sign of the determinant and orientation       109         14.4 Further examples       109         15 Eigenvalues and eigenvectors       111         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114         15.4 Figenvectors corresponding to distinct eigenvalues are linearly independent 115	14.2 The determinant $\ldots$	10
14.4 Further examples       109         15 Eigenvalues and eigenvectors       111         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114         15.4 Figenvectors commence of instance of the distinct eigenvalues are linearly independent 115	14.3 Sign of the determinant and orientation	10
15 Eigenvalues and eigenvectors       111         15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114         15.4 Figenvectors common and ing to distinct eigenvalues are linearly independent 115	14.4 Further examples	10
15.1 Non-singular matrices       111         15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114         15.4 Eigenvectors commence and ing to distinct eigenvelves are linearly independent 115	15 Eigenvalues and eigenvectors	11 <sup>-</sup>
15.2 Eigenvalues and eigenvectors       112         15.2.1 Example       113         15.3 Simple properties       114         15.4 Figenvectors common onding to distinct eigenvalues are linearly independent 115	15.1 Non-singular matrices	11
15.2.1 Example       113         15.3 Simple properties       114         15.4 Figenvectors corresponding to distinct sigenvalues are linearly independent 115	15.2 Eigenvalues and eigenvectors	11
15.3 Simple properties	15.2.1 Example	11
15.4 Figenvectors conversed ing to distinct signary lung are linearly independent 115	15.3 Simple properties	11.
10.4 FAPERVECTORS COLLESDORITING TO CUSPINCE EIGENVALUES ALE HIERALIV INCEDENCEMENTS	15.4 Eigenvectors corresponding to distinct eigenvalues are linearly independent	

16	Diag	gonalisation	117
	16.1	Similar matrices	118
		16.1.1 Theorem (similar matrices)	118
	16.2	A closer look at the diagonal matrix	118
	16.3	Diagonalisability	119
		16.3.1 Example	119
	16.4	Algebraic and geometric multiplicity	121
	16.5	Applications of diagonalisability	122
		16.5.1 Systems of differential equations	122
		16.5.2 Matrix powers	123
		16.5.3 Linear systems	123
17	Ortl	hogonal matrices	125
	17.1	Transpose of a transition matrix	125
	17.2	Orthogonal matrices	127
	17.3	Isometries	128
	17.4	Orthogonal matrices generate isometries $\ldots \ldots \ldots \ldots \ldots \ldots$	129
18	Ortl	hogonal diagonalisation	131
	18.1	Orthogonal matrices	131
	18.2	Symmetric matrices	131
		18.2.1 If A is real symmetric, then the eigenvectors corresponding to	
		different eigenvalues are orthogonal	132
		18.2.2 Real symmetric matrices are orthogonally diagonalisable	133
		18.2.3 Eigenvectors and eigenvalues	133
		18.2.4 Example	134
19	Qua	dratic forms	136
	19.1	Definition	136
		19.1.1 Give the matrix representation of the quadratic form $2x^2 + 6xy - 7x^2 + 7x^2 $	107
	10.0	$(y^2, \ldots, \ldots,$	137
	19.2	Diagonalising quadratic forms $\dots \dots \dots$	199
		19.2.1 Express $-5x - 2y - 5z + 2xy + 2yz$ exclusively as a sum of	190
	10.2	Quadratia equations and conic sections	100
	19.5	Quadratic equations and come sections $\dots \dots \dots$	139
	10.4	19.5.1 Describe the conic whose equation is $x^{2} + y^{2} + 2xy - 5x - 5y + 4 = 0$ Further reading: quadric surfaces	140
	19.4	Further reading. quadric surfaces	142
<b>20</b>	Con	nplex matrices	144
	20.1	Definition (conjugate transpose)	144
		20.1.1 Example	144
	20.2	Unitary matrices	145
	20.3	Complex inner product	145
	20.4	Hermitian (self-adjoint) matrices	146
		20.4.1 Proof that Hermitian matrices have real eigenvalues	146

	20.5	Unitary diagonalisation	147
	20.6	Normal matrices	147
		20.6.1 Example	148
	20.7	Normal = unitarily diagonalisable	149
		20.7.1 Example	149
21	Mu	ltivariable Taylor series	152
	21.1	Hessian matrix	152
	21.2	Example: <i>H</i> of $f(x, y) = x^3y + 2y$ at (1, 2)	152
	21.3	Taylor series of $f(x,y)$	153
	21.4	Example: $f(x, y) = \frac{\sin(x-y^2)}{2}$ about (0, 0)	155
	21.1	Taylor series in any dimension $\dots \dots \dots$	155
იი	0	ical mainta in malina mainna	150
22	Orit 201	Classification of critical points in a dimensions	150
	22.1	Classification of critical points in $n$ dimensions	150
	22.2	Critical points by Taylor gapies	157
	22.3 22.4	Example: $Q = ax^2 + bxy + cy^2$	150
	22.4	Example. $Q = ax + bxy + cy \dots $	109
<b>23</b>	Intr	oduction to double integrals, volume below a surface	162
	23.1	Double integrals	162
	23.2	Properties of the double integral	164
	23.3	Iterated integrals	165
		23.3.1 Example: evaluate $\int_{0}^{2} \int_{1}^{3} x^2 y  dy  dx$	165
		23.3.2 Example: evaluate $\int_1^3 \int_0^2 x^2 y  dx  dy$	166
24	Fub	ini's theorem, volume by slabs	168
- 1	24.1	Fubini's theorem	168
	24.2	Example: evaluate $\iint_{R} (x^2 + y^2) dA$ where	100
		$R = \{(x, y)   0 \le x \le 2, \ 0 \le y \le 1\}$	168
	24.3	Interpreting Fubini's theorem in terms of volume	170
	24.4	Example: find the volume of the solid bounded by the elliptic paraboloid	
		$x^{2} + 2y^{2} + z = 16$ , the planes $x = 2$ and $y = 2$ , and the three coordinate	
		planes	171
	24.5	Special case when $f(x, y) = g(x)h(y)$	173
		24.5.1 Example: $\iint_R \sin x \cos y  dA \text{ where } R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \dots \dots \dots$	173
25	Inte	grals over general regions	175
-	25.1	Type I regions	176
		25.1.1 Example: find $\iint_{D} (4x + 10y) dA$ where D is the region between	. •
		the parabola $y \stackrel{D}{=} x^2$ and the line $y = x + 2$	177

the line $y = x - 1$ and the parabola $y^2 = 2x + 6$		25.2	Type II regions $\dots \dots \dots$	179
integral $\iint_{D} f(x, y) dA$ , for some integrable function $f$		25.3	the line $y = x - 1$ and the parabola $y^2 = 2x + 6$ Express $D$ as a union of regions of type I or type II and expand the	180
26 Interchanging order of integration1826.1 Find the volume under the paraboloid $z = x^2 + y^2$ above the region $D$ , where $D$ is bounded by $y = x^2$ and $y = 2x$ . Do the problem twice, first by taking $D$ to be a type I region, then by taking $D$ to be type II.26.2 Example: Find $\int_0^1 \int_x^1 \sin(y^2) dy dx$ 1827.1 Example: Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2, x = 2y, x = 0$ and $z = 0$ .1927.2 Area1927.3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 1928 Double integrals in polar coordinates $D$ for $e^{-(x^2+y^2)} dx dy$ where $D$ is the region bounded by the circle $x^2 + y^2 = R^2$ .1928.2 Example: Find $\iint_D e^{-(x^2+y^2)} dx dy$ where $D$ is the region bounded by the circle $x^2 + y^2 = R^2$ .2028.3 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$ .2028.3 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 2x$ , for $z \ge 0$ .2029 Mass, centre of mass and moments 29.1 Example: find the centre of mass of a rectangle with vertices $(0,0), (1,0)$ and $(0,2)$ with constant density $\rho_0$ .2029.2 Example: find the centre of mass of a rectangle with vertices $(0,0), (2,0),$ $(2,1)$ and $(0,1)$ with density $p(x,y) = 6x + 12y$ .2130.1 Find the mass of a rectangular block with dimensions $0 \le x \le L$ , $0 \le y \le W$ and $0 \le z \le H$ if the density is $\rho = \rho_0 + \alpha xyz$ .2130.2 Evaluat $\iiint_x dV$ over the region $R$ bounded by the surfaces $x = 0$ , $y = 0, z = 0$ and $x + y + z = 1$ .2130.3 Changing the order of integration21			integral $\iint_D f(x,y)  dA$ , for some integrable function $f. \ldots \ldots$	182
26.1 Find the volume under the paraboloid $z = x^2 + y^2$ above the region $D$ , where $D$ is bounded by $y = x^2$ and $y = 2x$ . Do the problem twice, first by taking $D$ to be a type I region, then by taking $D$ to be type II 18 26.2 Example: Find $\int_0^1 \int_x^1 \sin(y^2)  dy  dx$	26	Inte	rchanging order of integration	184
26.2 Example: Find $\int_{0}^{1} \int_{x}^{1} \sin(y^{2}) dy dx$		26.1	Find the volume under the paraboloid $z = x^2 + y^2$ above the region $D$ , where $D$ is bounded by $y = x^2$ and $y = 2x$ . Do the problem twice, first by taking $D$ to be a type I region, then by taking $D$ to be type II	184
27 Review of applications: volume, area1827.1 Example: Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2, x = 2y, x = 0$ and $z = 0. \dots $		26.2	Example: Find $\int_0^1 \int_x^1 \sin(y^2)  dy  dx$	186
27.1 Example: Find the volume of the tetrahedron bounded by the planes x + 2y + z = 2, x = 2y, x = 0 and $z = 0.$	<b>27</b>	Rev	iew of applications: volume, area	189
$x + 2y + z = 2, x = 2y, x = 0 \text{ and } z = 0. \dots $		27.1	Example: Find the volume of the tetrahedron bounded by the planes	
27.2 Area		97.9	$x + 2y + z = 2, x = 2y, x = 0$ and $z = 0. \dots \dots \dots \dots$	190
27.3 Find the area enclosed by the empse $\frac{a^2}{a^2} + \frac{b^2}{b^2} = 1$		21.2	Find the area enclosed by the ellipse $\frac{x^2}{y^2} + \frac{y^2}{1}$	192
<b>28 Double integrals in polar coordinates19</b> 28.1 Example: Find $\iint_{D} e^{-(x^2+y^2)} dx dy$ where $D$ is the region bounded by the circle $x^2 + y^2 = R^2$ 1928.2 Example: Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$ 2028.3 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 2x$ , for $z \ge 0$ .20 <b>29 Mass, centre of mass and moments20</b> 29.1 Example: find the centre of mass of a triangular lamina with vertices $(0, 0), (1, 0)$ and $(0, 2)$ with constant density $\rho_0$ .2029.2 Example: find the centre of mass of a rectangle with vertices $(0, 0), (2, 0),$ $(2, 1)$ and $(0, 1)$ with density $\rho(x, y) = 6x + 12y$ .21 <b>30 Introduction to triple integrals21</b> 30.1 Find the mass of a rectangular block with dimensions $0 \le x \le L,$ $0 \le y \le W$ and $0 \le z \le H$ if the density is $\rho = \rho_0 + \alpha xyz$ .2130.3 Changing the order of integration21 <b>31 Cylindrical coordinates21</b>		21.5	Find the area enclosed by the empse $\frac{1}{a^2} + \frac{1}{b^2} = 1$	195
28.1 Example: Find $\iint_{D} e^{-(x^2+y^2)} dx dy$ where <i>D</i> is the region bounded by the circle $x^2 + y^2 = R^2$	28	Dou	ble integrals in polar coordinates	196
the circle $x^2 + y^2 = R^2$		28.1	Example: Find $\iint e^{-(x^2+y^2)} dx dy$ where D is the region bounded by	
28.2 Example: Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$			the circle $x^2 + y^2 = B^2$	198
the paraboloid $z = 1 - x^2 - y^2$		28.2	Example: Find the volume of the solid bounded by the plane $z = 0$ and	100
28.3 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 2x$ , for $z \ge 0$			the paraboloid $z = 1 - x^2 - y^2$	201
29 Mass, centre of mass and moments2029.1 Example: find the centre of mass of a triangular lamina with vertices $(0,0), (1,0)$ and $(0,2)$ with constant density $\rho_0$		28.3	Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and inside the cylinder $x^2 + y^2 - 2x$ for $z \ge 0$	202
<b>29 Mass, centre of mass and moments20</b> 29.1 Example: find the centre of mass of a triangular lamina with vertices $(0,0), (1,0)$ and $(0,2)$ with constant density $\rho_0$ .2029.2 Example: find the centre of mass of a rectangle with vertices $(0,0), (2,0),$ $(2,1)$ and $(0,1)$ with density $\rho(x,y) = 6x + 12y$ .21 <b>30 Introduction to triple integrals21</b> $30.1$ Find the mass of a rectangular block with dimensions $0 \le x \le L,$ 			and inside the cylinder $x + y = 2x$ , for $z \ge 0$	202
29.1 Example: find the centre of mass of a triangular lamina with vertices $(0,0), (1,0)$ and $(0,2)$ with constant density $\rho_0$	29	Mas	s, centre of mass and moments	205
29.2 Example: find the centre of mass of a fectial gie with vertices $(0, 0), (2, 0), (2, 1)$ and $(0, 1)$ with density $\rho(x, y) = 6x + 12y. \dots \dots \dots 21$ 30 Introduction to triple integrals 21 30.1 Find the mass of a rectangular block with dimensions $0 \le x \le L, 0 \le y \le W$ and $0 \le z \le H$ if the density is $\rho = \rho_0 + \alpha xyz. \dots \dots 21$ 30.2 Evaluate $\iiint_R z \ dV$ over the region $R$ bounded by the surfaces $x = 0, y = 0, z = 0$ and $x + y + z = 1. \dots \dots \dots \dots \dots \dots 21$ 30.3 Changing the order of integration $\dots \dots \dots$		29.1	Example: find the centre of mass of a triangular lamina with vertices $(0,0), (1,0)$ and $(0,2)$ with constant density $\rho_0$	208
<b>30 Introduction to triple integrals21</b> $30.1$ Find the mass of a rectangular block with dimensions $0 \le x \le L$ , $0 \le y \le W$ and $0 \le z \le H$ if the density is $\rho = \rho_0 + \alpha xyz$ . <b>21</b> $30.2$ Evaluate $\iiint_R z  dV$ over the region R bounded by the surfaces $x = 0$ , $y = 0, z = 0$ and $x + y + z = 1$ . <b>21</b> $30.3$ Changing the order of integration <b>2131 Cylindrical coordinates21</b>		29.2	(2,1) and (0,1) with density $\rho(x,y) = 6x + 12y$	210
30.1 Find the mass of a rectangular block with dimensions $0 \le x \le L$ , $0 \le y \le W$ and $0 \le z \le H$ if the density is $\rho = \rho_0 + \alpha x y z$ .2130.2 Evaluate $\iiint_R z \ dV$ over the region R bounded by the surfaces $x = 0$ , $y = 0, z = 0$ and $x + y + z = 1$ .2130.3 Changing the order of integration21 <b>31 Cylindrical coordinates21</b>	30	Intr	oduction to triple integrals	212
$0 \le y \le W \text{ and } 0 \le z \le H \text{ if the density is } \rho = \rho_0 + \alpha xyz. \dots 21$ $30.2 \text{ Evaluate } \iiint_R z \ dV \text{ over the region } R \text{ bounded by the surfaces } x = 0,$ $y = 0, z = 0 \text{ and } x + y + z = 1. \dots \dots \dots \dots \dots \dots \dots 21$ $30.3 \text{ Changing the order of integration } \dots $		30.1	Find the mass of a rectangular block with dimensions $0 \le x \le L$ ,	010
$y = 0, z = \overset{R}{0}$ and $x + y + z = 1.$		30.2	$0 \le y \le W$ and $0 \le z \le H$ if the density is $\rho = \rho_0 + \alpha x y z$ Evaluate $\iiint_R z  dV$ over the region R bounded by the surfaces $x = 0$ ,	213
30.3 Changing the order of integration			$y = 0, z = {\stackrel{\kappa}{0}}$ and $x + y + z = 1.$	214
31 Cylindrical coordinates 21		30.3	Changing the order of integration	215
J J	<b>31</b>	Cyli	ndrical coordinates	217

	31.1	A simple example: Find the volume of a cylinder of radius $R$ and height $H$ . (Ans. $\pi R^2 H$ )	219
	31.2	Find the mass of the solid defined by the region contained within the cylinder $x^2 + y^2 = 1$ below the plane $z = 4$ and above the paraboloid $z = 1$	-
		$1 - x^2 - y^2$ . The density at any given point in the region is proportional to the distance from the axis of the cylinder	220
<b>32</b>	$\mathbf{Sph}$	erical coordinates	223
	32.1	A simple example: Find the volume of a sphere of radius $R$	225
	32.2	Find the mass of a sphere of radius $R$ whose density is given by $\rho(x,y,z) =$	$e^{-(x^2+y^2+z^2)^{1/2}}.226$
	32.3	Find the volume of the "ice cream cone" $R$ between a sphere of radius	
		a (centred at the origin) and the cone $z = \sqrt{x^2 + y^2}$	227
33	Mor	nents of inertia (second moments)	230
	33.1	Derive the integral formula for $I_x$	231
	33.2	Example: locate the centre of mass of a solid hemisphere of radius $a$ with density proportional to the distance from the centre of the base.	
		Find its moment of inertia about the z-axis.	232
34	Con	servative vector fields	235
	34.1	Vector fields	235
		34.1.1 Example: $F(r) = (-y, x) = -yi + xj$	235
		34.1.2 Example: Newtonian gravitational field	236
	34.2	Gradient of a scalar field, conservative vector fields $\ \ldots \ \ldots \ \ldots \ \ldots$	237
		34.2.1 Example: find the gradient of $f(x, y, z) = x^2 y^3 z^4$	237
		34.2.2 Verify that the Newtonian gravitational field is conservative with	
		potential function $f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$	238
		34.2.3 The vector field $\mathbf{F}(x,y) = (3+2xy)\mathbf{i} + (x^2-3y^2)\mathbf{j}$ is conservative.	
		Find a corresponding potential function. $\dots$ $3^{2}$	239
		34.2.4 The vector field $\mathbf{F}(x, y, z) = y^2 \mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$ is con-	240
		servative. Find a corresponding potential function.	240
35	The	fundamental theorem for line integrals, path independence	<b>242</b>
	35.1	Line integrals in the plane	242
	35.2	Work done by a force, line integrals of vector fields	243
	35.3	Evaluating line integrals of vector fields	244
	35.4	Common notation $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$	244
		35.4.1 Example: let $A = (0, 1), B = (1, 2)$ . Evaluate $\int_C ((x^2 - y)dx + (y^2 - y)dx) dx$	(+x)dy
		along the curve $C$ given by: (i) the straight line from $A$ to $B$ ;	- 1 <b>-</b>
	05 5	(11) the parabola $y = x^2 + 1$ from A to B	245
	35.5	Line integrals of conservative vector fields, path independence	247
		35.5.1 The fundamental theorem for line integrals	247

36	Gre	en's th	eorem and a test for conservative fields	252
	36.1	The st	ory so far	252
	36.2	Clairau	ut's theorem and consequences	252
	36.3	Green'	s theorem	254
		36.3.1	Find the work done by the force $F = x^2 y i + x y^2 j$ anticlockwise	
			around the circle with centre at the origin and radius $a. \ldots$	255
		36.3.2	Evaluate the line integral $\int_C 2xy \ dx + (x^2 + 3y^2) \ dy$ , where C is	
			the path from $(0,1)$ to $(1,0)$ along $y = (x-1)^2$ and then from $(1,0)$ to $(2,1)$ along $y = x - 1 \dots \dots \dots \dots \dots \dots \dots \dots \dots$	256
		36.3.3	Evaluate $\int (3+2xy)dx + (x^2-3y^2)dy$ where C is the curve parameter.	netrised
			by $\boldsymbol{r}(t) = (1 - \cos(\pi t))\boldsymbol{i} + (1 + \sin^3(\pi t))\boldsymbol{j}$ for $0 \le t \le 1/2$	257
37	On	provin	g Green's theorem	259
	37.1	Region	of type I	260
	37.2	Region	of type II	262
	37.3	Region	of both type I and II	263
	37.4	More g	general regions	263
38	Flux	x of a v	vector field	266
	38.1	Flux in	n 2D	268
		38.1.1	Evaluating flux in 2D	270
		38.1.2	Calculate the flux of $\boldsymbol{v} = -y\boldsymbol{i} + x\boldsymbol{j}$ (in the positive x direction)	
			across the line $x = 2$ (for $2 \le y \le 6$ )	271
	38.2	Outwa	rd flux across a closed curve in the plane	273
		38.2.1	Calculate the outward flux of $\boldsymbol{v} = xy\boldsymbol{i} + xy\boldsymbol{j}$ across the curve	
			from $(2,0)$ to $(-2,0)$ via the semicircle of radius 2 centred at the	
			origin (for $y \ge 0$ ) followed by the straight line from (-2,0) to (2,0)	.273
39	Dive	ergence	e of a vector field (div)	276
	39.1	Calcul	ating divergence	276
		39.1.1	Example: $\boldsymbol{v} = xy^2\boldsymbol{i} + xyz\boldsymbol{j} + yz^2\boldsymbol{k}$ . Find div $\boldsymbol{v}$	276
	39.2	Unders	standing div in two dimensions	277
	39.3	Outwa	rd flux across a closed curve in the plane (revisited) $\ldots \ldots$	279
	39.4	Relatio	onship to Green's theorem	280
		39.4.1	Use the flux form of Green's theorem to calculate the outward	
			flux of $\boldsymbol{v} = xy\boldsymbol{i} + xy\boldsymbol{j}$ across the curve from (2,0) to (-2,0) via the	
			semicircle of radius 2 centred at the origin (for $y \ge 0$ ) followed	
			by the straight line from $(-2,0)$ to $(2,0)$	281
		39.4.2	For the following graphs of vector fields, determine whether the	
			divergence is positive, negative or zero	282
40	Para	ametri	sation of surfaces in $\mathbb{R}^3$	285
	40.1	Param	etric surfaces	285
	40.2	Param	etrising surfaces using cylindrical and spherical coordinates	286

		$   \begin{array}{l}     40.2.1 \\     40.2.2   \end{array} $	Parametrise the paraboloid $z = 1 - x^2 - y^2$ for $z \ge 0$ Parametrise the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = 2$ and $z = -2$	288 290
	40.3	Tanger	t planes $\ldots$	290 291
		40.3.1	Find the tangent plane to the surface parametrised by $\mathbf{r}(u, v) = u^2 \mathbf{i} + v^2 \mathbf{j} + (u + 2v) \mathbf{k}$ at the point $(1, 1, 3)$	292
41	Surf	face int	tegrals	<b>294</b>
	41.1	Area o	f a parametric surface	294
		41.1.1	Application: find the surface area of the paraboloid $z = 1 - x^2 - y^2$ for $z \ge 0$ .	297
	41.2	More c 41.2.1	on calculating surface integrals, applications	298
			any point $(x, y, z)$ on the surface of a solid hemisphere of radius	
			I centred at the origin, defined for $z \ge 0$ . Find the average temperature over the surface.	299
42	Vari	able tr	cansformations in double integrals	302
	42.1	Two-va	ariable change of coordinates	302
	42.2	Find the	he image of the region $\{(u, v) \mid 0 \le u, v \le 1\}$ under the transfor-	202
	42.3	Ination Jacobi	$x = u^2 - v^2, y = 2uv \dots \dots$	303 304
	42.4	Examp	ble: $R = \{(x, y) \mid 0 \le x \le 1, x^2 \le y \le x\}$	306
43	Fluz	c integ	rals and Gauss' divergence theorem	308
	43.1	Orienta	able surfaces	308
	43.2	The flu 43.2.1	IX integral	308
			the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le (x, y, z) \}$	
			the surface of the cylindrical solid given by $\{(x, y, z) \mid x^2 + y^2 \le 1, 0 \le z \le 2\}$ .	310
	43.3	Gauss' 43.3.1	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem	310 312
	43.3	Gauss' 43.3.1	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$	310 312 313
44	43.3 On :	Gauss' 43.3.1 proving	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ g Gauss' divergence theorem	<ul><li>310</li><li>312</li><li>313</li><li>315</li></ul>
44	43.3 <b>On</b> 44.1	Gauss' 43.3.1 proving Special	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = zi + yj + xk$ across the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ <b>g</b> Gauss' divergence theorem case proof	310 312 313 <b>313</b> <b>315</b>
44 45	43.3 <b>On</b> 44.1 <b>Cur</b>	Gauss' 43.3.1 proving Special l of a v	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = zi + yj + xk$ across the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ <b>g Gauss' divergence theorem</b> case proof	<ul> <li>310</li> <li>312</li> <li>313</li> <li>315</li> <li>319</li> </ul>
44 45	43.3 <b>On</b> 44.1 <b>Cur</b> 45.1	Gauss' 43.3.1 proving Special l of a v Calcula	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = zi + yj + xk$ across the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ <b>g Gauss' divergence theorem</b> l case proof	310 312 313 <b>315</b> 315 <b>319</b> 319
44 45	43.3 On 44.1 Cur 45.1	Gauss' 43.3.1 proving Special l of a v Calcula 45.1.1	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ <b>g Gauss' divergence theorem</b> l case proof	310 312 313 <b>313</b> <b>315</b> <b>315</b> <b>319</b> 319 319 320
44 45	43.3 On 44.1 Cur 45.1 45.2 45.2	Gauss' 43.3.1 proving Special l of a v Calcula 45.1.1 Unders Conser	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ <b>g Gauss' divergence theorem</b> l case proof <b>vector field</b> ating curl Example: let $\mathbf{v} = yz^2\mathbf{i} + zx^2\mathbf{j} + xy^2\mathbf{k}$ . Find curl $(\mathbf{v})$ standing curl	310 312 313 <b>315</b> 315 <b>319</b> 319 319 320 321
44 45	43.3 On 44.1 Cur 45.1 45.2 45.3	Gauss' 43.3.1 proving Special l of a v Calcula 45.1.1 Unders Conser 45.3.1	the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ divergence theorem Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the surface of the cylindrical solid given by $\{(x, y, z)   x^2 + y^2 \le 1, 0 \le z \le 2\}$ <b>g Gauss' divergence theorem</b> l case proof <b>vector field</b> ating curl Example: let $\mathbf{v} = yz^2\mathbf{i} + zx^2\mathbf{j} + xy^2\mathbf{k}$ . Find curl $(\mathbf{v})$ standing curl	310 312 313 <b>315</b> 315 <b>319</b> 319 320 321

46 Stokes' theorem	<b>324</b>
46.1 Summary of surfaces and curves	324
$46.1.1 \text{ Surfaces } \ldots $	324
$46.1.2  \text{Curves}  \ldots  \ldots  \ldots  \ldots  \ldots  \ldots  \ldots  \ldots  \ldots  $	324
46.2 Stokes' theorem $\ldots$	325
46.2.1 Relation to Green's theorem	325
46.2.2 Verify Stokes' theorem where $C$ is the curve of intersection of	
the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ , oriented	
counterclockwise when looking from above, and $\mathbf{F} = [-y^2, x, z^2]$ .	325
46.3 Further reading: Circulation	327
46.4 Further reading: Curl fields and vector potentials	328

# 1 Solutions of first order ODEs

By the end of this section, you should be able to answer the following questions about first order ODEs:

- How do you solve an IVP associated with directly integrable, separable or linear ODEs? (Revision)
- Under what conditions does a solution to an IVP problem exist?
- Under what conditions is a solution to an IVP problem unique?

In MATH1052, you were introduced to Ordinary Differential Equations (ODEs) and Initial Value Problems (IVPs) and saw how to find solutions to some special types of first order equations. In particular, there should be three types of first order ODEs that you are familiar with solving.

• Directly integrable:  $\frac{dy}{dx} = f(x)$ .

• Separable:  $\frac{dy}{dx} = f(x)g(y)$ .

• Linear: 
$$\frac{dy}{dx} = q(x) - p(x)y$$
.

In most applications involving first order ODEs, we are required to solve an IVP. Generally, this is a problem of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

In other words, we seek to find solutions of the ODE which pass through the point  $(x_0, y_0)$  in the x-y plane.

Consider the following three examples.

# **1.1 Example:** $\frac{dy}{dx} = x$ , y(0) = 1 has a unique solution

# 1.2 Example: $\frac{dy}{dx} = 3xy^{1/3}$ , y(0) = 0 has more than one solution

**1.3 Example:** 
$$\frac{dy}{dx} = \frac{x-y}{x}$$
,  $y(0) = 1$  has no solution

#### 1.4 Existence and uniqueness criteria

Here we consider the initial value problem of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

The main result concerns the conditions under which we have existence and uniqueness of a solution.

If f and  $f_y$  are continuous in some rectangle  $R = \{(x, y) | |x - x_0| \le a, |y - y_0| \le b\},$ then there is some interval  $|x - x_0| \le h \le a$  which contains a unique solution  $y = \phi(x)$  of the initial value problem  $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$ 

The above theorem only tell us that a solution exists or is unique locally (i.e., in the rectangle R). Beyond R, we simply don't know. Let's look at the previous three examples in the context of the theorem. In the next lecture, we look closer at understanding these conditions and how they arise.

**1.5 Example:**  $\frac{dy}{dx} = x, \ y(0) = 1$ 

**1.6 Example:** 
$$\frac{dy}{dx} = 3xy^{1/3}, y(0) = 0$$

**1.7 Example:** 
$$\frac{dy}{dx} = \frac{x-y}{x}, y(0) = 1$$

Notes.

## 2 Method of successive approximations

By the end of this section, you should be able to answer the following questions:

- How do you convert a first order ODE to an integral equation?
- How do you generate the sequence of functions for method of successive approximations?
- Under what conditions does this iterative method work? (Or: What can go wrong?)

Here we look at a proof of the existence and uniqueness theorem from the previous chapter. The presentation is based on that of the book *Elementary Differential Equations and Boundary Value Problems* by W.E. Boyce and R.C. DiPrima (ed. 10, Wiley, 2012).

#### 2.1 A modified theorem

To start, we note that it is always possible to apply a variable shift so that the initial value problem can be written as

$$\frac{dy}{dx} = f(x, y), \quad y(0) = 0.$$
 (1)

#### **2.1.1 Example:** y' = 2(x - 1)(y - 1), y(1) = 2

Without loss of generality, we will consider this problem where the initial point is at the origin. We can restate the result as follows.

If f and  $f_y$  are continuous in some rectangle

$$R = \{ (x, y) | |x| \le a, |y| \le b \}$$

then there is some interval  $|x| \leq h \leq a$  which contains a unique solution  $y = \phi(x)$  of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(0) = 0$$

#### 2.2 Equivalence with integral equation

Let  $y = \phi(x)$  be a solution to the IVP (1), and note that the function  $F(x) = f(x, \phi(x))$  is a continuous function of x only. We then have

$$\phi(x) = \int_0^x F(t) \, dt = \int_0^x f(t, \phi(t)) \, dt.$$
(2)

Note that  $\phi(0) = 0$ . This is an example of an *integral equation*. Conversely, let  $\phi(x)$  satisfy the integral equation (2). By the Fundamental Theorem of Integral Calculus,  $\phi'(x) = f(x, \phi(x))$ , which implies that  $y = \phi(x)$  is a solution of the IVP (1). In other words, the IVP (1) and the integral equation (2) are *equivalent*, meaning that a solution of one is a solution of the other. Herein we work with (2).

#### 2.3 Method of successive approximations

The goal of the approach is to generate a sequence of functions  $\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\}$ . Starting with the initial function  $\phi_0(x) = 0$  (satisfying the initial condition of (1)), the sequence is generated iteratively by

$$\phi_{n+1}(x) = \int_0^x f(t, \phi_n(t)) \, dt.$$
(3)

Note that each  $\phi_n$  satisfies  $\phi_n(0) = 0$ , but generally not the integral equation (2) itself. If there is a k, however, such that  $\phi_{k+1}(x) = \phi_k(x)$ , then  $\phi_k(x)$  is a solution of the integral equation (2) and hence the IVP (1). Generally this does not occur, but we may instead consider *limit functions*.

There are four key points to consider:

- 1. Do all members of the sequence *exist*?
- 2. Does the sequence *converge* to a limit function  $\phi$ ?
- 3. What are the properties of  $\phi$ ?
- 4. If  $\phi$  satisfies the IVP (1), are there other solutions?

**2.4 Example:** y' = 2x(y+1), y(0) = 0

#### 2.5 Discussion

Even though the example demonstrates many of the features that need to be considered, here we elaborate more on the four points outlined earlier.

1. Existence of  $\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\}$ 

The conditions of the theorem state that f and  $f_y$  are continuous in the rectangle. Even still, it may be that there is a j such that the curve  $y = \phi_j(x)$  contains points outside R. In such a case, to apply the iteration and determine  $\phi_{j+1}(x)$ , we would have to evaluate f(x, y) at points where f may not be continuous or even defined! We can, however, restrict x to a smaller interval if necessary.

To find such an interval, we note that if f is continuous on a closed region, then f is bounded. In other words, there is an M such that

$$|f(x,y)| \le M, \quad \forall (x,y) \in R.$$

Also,  $\phi_j(0) = 0$ . Since  $f(x, \phi_j(x)) = \phi'_{j+1}(x)$ , the largest slope of the curve  $y = \phi_{j+1}(x)$  is M. There are two situations which can arise, as depicted in the following diagrams:



We can therefore say that  $(x, \phi_{k+1}(x)) \in R$  provided  $|x| \leq h$ , where  $h = \min(\frac{b}{M}, a)$ . We then consider the values of (x, y) in the (possibly smaller) rectangle

$$R' = \{ (x, y) | |x| \le h, |y| \le b \}.$$

In this case, all members of the sequence exist.

#### 2. Convergence?

Note that we can write

$$\phi_n(x) = \phi_1(x) + [\phi_2(x) - \phi_1(x)] + [\phi_3(x) - \phi_2(x)] + \dots + [\phi_n(x) - \phi_{n-1}(x)], \quad (4)$$

which is the nth partial sum of the series

$$S(x) = \phi_1(x) + \sum_{k=1}^{\infty} [\phi_{k+1}(x) - \phi_k(x)].$$

Convergence of S(x) implies convergence of the sequence  $\{\phi_n(x)\}$ .

Using the Mean Value Theorem, it is possible to show that if  $f_y$  is continuous in R', then there exists a K > 0 such that

$$|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|,$$

where  $(x, y_1), (x, y_2) \in R'$ . In particular, we have

$$|f(x,\phi_n(x)) - f(x,\phi_{n-1}(x))| \le K |\phi_n(x) - \phi_{n-1}(x)|.$$
(5)

Set n = 1 in (5), and show that  $\exists M > 0$  such that  $|\phi_2(x) - \phi_1(x)| \le MK|x|^2/2$ .

It is then possible to show by induction that

$$|\phi_n(x) - \phi_{n-1}(x)| \le \frac{MK^{n-1}|x|^n}{n!} \le \frac{MK^{n-1}h^n}{n!}.$$

Given the expression (4) above for  $\phi_n(x)$ , we have

$$\begin{aligned} |\phi_n(x)| &\leq |\phi_1(x)| + |\phi_2(x) - \phi_1(x)| + \dots + |\phi_n(x) - \phi_{n-1}(x)| \\ &\leq \frac{M}{K} \left( Kh + \frac{(Kh)^2}{2!} + \dots + \frac{(Kh)^n}{n!} \right) \end{aligned}$$
(6)

In the limit as  $n \to \infty$ , this expression becomes

$$\frac{M}{K}(e^{Kh}-1).$$

Therefore the sum in (6) converges, and so the sum in (4) is absolutely convergent. We conclude that the sequence  $\{\phi_n(x)\}$  converges, since it is the sequence of partial sums of a convergent series.

3. Properties of  $\phi(x) = \lim_{n \to \infty} \phi_n(x)$ 

We can make use of the previous discussion in Point 2 to establish *uniform convergence* (see MATH2400/MATH2401) of the sequence  $\{\phi_n\}$ . In particular, the functions  $\phi_n$  are continuous for all n, so uniform convergence implies the limit function is also continuous.

Now we take the limit of both sides of equation (3):

$$\phi(x) = \lim_{n \to \infty} \int_0^x f(s, \phi_n(s)) ds$$
$$= \int_0^x \lim_{n \to \infty} f(s, \phi_n(s)) ds.$$

This last step is allowed only because the sequence  $\{\phi_n\}$  converges uniformly. Also, since f is continuous, we can develop this as

$$\phi(x) = \int_0^x f(s, \lim_{n \to \infty} \phi_n(s)) ds$$
$$= \int_0^x f(s, \phi(s)) ds.$$

That is,  $\phi$  solves the integral equation (2) and hence the IVP (1).

#### 4. Uniqueness?

Here we proceed as in the earlier example. That is, we assume  $y = \psi(x)$  is another solution. It is possible to show that

$$|\phi(x) - \psi(x)| \le A \int_0^x |\phi(s) - \psi(s)| ds$$

for  $0 \le x \le h$  and suitable A > 0. Then apply the same argument as in the example.

Notes.

### 3 Exact first order ODEs

By the end of this section, you should be able to answer the following questions about first order ODEs:

- How do you identify an exact ODE?
- How do you solve an exact ODE?

#### 3.1 Definition

First recall that if z = f(x, y) is a differentiable function of x and y, where x = g(t)and y = h(t) are both differentiable functions of t, then z is a differentiable function of t whose derivative is given by the chain rule:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Now suppose the equation

$$f(x,y) = C$$

defines y implicitly as a function of x (here C is a constant). Then y = y(x) can be shown to satisfy a first order ODE obtained by using the chain rule above. In this case, z = f(x, y(x)) = C, so

$$(0 = ) \frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$
  

$$\Rightarrow f_x + f_y y' = 0.$$
(7)

A first order ODE of the form

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$
(8)

is called exact if there is a function f(x, y) (compare (8) with (7) above) such that

$$f_x(x,y) = P(x,y)$$
 and  $f_y(x,y) = Q(x,y)$ .

The solution is then given implicitly by the equation

$$f(x,y) = C.$$

The constant C can usually be determined by some kind of "initial condition".

Given an equation of the form (8), how do we determine whether or not it is exact? There is a simple test.

### **3.2** Test for exactness

Let 
$$P, Q, \frac{\partial P}{\partial y}$$
, and  $\frac{\partial Q}{\partial x}$  be continuous over some region of interest. Then  

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0$$
is an exact ODE iff  

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

everywhere in the region.

The problem of actually determining f(x, y) is still outstanding. Consider the following example.

# **3.3 Example:** $2x + e^y + xe^y y' = 0$

#### **3.4** Almost exact ODEs and integrating factors

Let's say that we have an equation

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

such that

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}.$$

The test we have just seen tells us that the ODE is not exact. Are we still able to do anything with it? Here we consider using an "integrating factor", which is different to the one introduced to solve linear ODEs.

The idea is to multiply the ODE by a function h(x, y) and then see if it is possible to choose h(x, y) such that the resulting equation

$$h(x,y)P(x,y) + h(x,y)Q(x,y)\frac{dy}{dx} = 0$$

is exact. We know from the test that this new equation is exact if and only if

$$\frac{\partial}{\partial y}(hP) = \frac{\partial}{\partial x}(hQ).$$

Let's see if we can find such a function:

In general, the equation for h(x, y) is usually just as difficult to solve as the original ODE. In some cases, however, we may be able to find an integrating factor which is a function of only one of the variables x or y. Let's try  $h \equiv h(x)$ :

**3.5 Example:**  $(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$ 



Notes.

# 4 Hyperbolic functions

By the end of this section, you should be able to answer the following questions:

- What is the definition of the sinh and cosh functions?
- What is the definition of the inverse hyperbolic functions?
- What are the derivatives and anti-derivatives of these functions?
- How are hyperbolic functions used in the catenary problem?

#### 4.1 **Properties of hyperbolic functions**

We define the functions  $\cosh(x)$  and  $\sinh(x)$  by

$$\cosh(x) = \frac{e^x + e^{-x}}{2},$$
  
 $\sinh(x) = \frac{e^x - e^{-x}}{2}.$ 

We can check by direct calculation that

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Compare this with the identity

$$\cos^2(\theta) + \sin^2(\theta) = 1 \tag{9}$$

for trig functions. The identity (9) allows us to parametrise a unit circle. By setting  $x(t) = \cos(t), y(t) = \sin(t)$ , we have

$$\cos^2(t) + \sin^2(t) = x^2 + y^2 = 1,$$

which is the equation of the unit circle.

If we set  $x(t) = \cosh(t)$  and  $y(t) = \sinh(t)$ , this gives a parametrisation for a hyperbola (only the right branch), since

$$\cosh^2(t) - \sinh^2(t) = x^2 - y^2 = 1,$$

which is the equation of a hyperbola. This is why we call these functions "hyperbolic functions".

These hyperbolic functions satisfy properties similar to their trigonometric counterparts. For example

$$\frac{d}{dx}(\cosh(x)) = \frac{e^x - e^{-x}}{2} = \sinh(x),$$
$$\frac{d}{dx}(\sinh(x)) = \frac{e^x + e^{-x}}{2} = \cosh(x).$$

 $\cosh(0) = 1$ ,  $\cosh(x) \ge 1$ ,  $\cosh(x)$  is an even function.  $\sinh(0) = 0$ ,  $\sinh(x)$  is an odd function. We also define

$$\begin{aligned} \tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{1 - e^{-2x}}{1 + e^{-2x}}, \quad |\tanh(x)| < 1, \\ \coth(x) &= \frac{\cosh(x)}{\sinh(x)}. \end{aligned}$$



Figure 1: Graph of  $\cosh(x)$ 



Figure 2: Graph of  $\sinh(x)$ 



Figure 3: Graph of tanh(x)



Figure 4: Graph of  $\operatorname{coth}(x)$ 

# 4.2 Inverse hyperbolic functions

The inverse function of cosh is denoted arcosh.

The inverse function of sinh is denoted arsinh.

The inverse function of tanh is denoted artanh.



Figure 5: Graph of  $\operatorname{arcosh}(x)$ 



Figure 6: Graph of  $\operatorname{arsinh}(x)$ 



Figure 7: Graph of  $\operatorname{artanh}(x)$ 

We have the following:

$$\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arsinh}(x) + c$$
$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arcosh}(x) + c, \ x > 1.$$

**4.2.1** Show that 
$$\frac{d}{dx}(\operatorname{arsinh}(x)) = \frac{1}{\sqrt{1+x^2}}$$
4.2.2	Evaluate the integrals	ſ	$\int \frac{dx}{dx}$ and $\int$		dx
		J	$\frac{1}{\sqrt{1+x^2}}$ and $\int$	anu J	$\sqrt{x^2-1}$



**4.2.3** Show that 
$$\frac{d}{dx}(\operatorname{artanh}(x)) = \frac{1}{1-x^2}, |x| < 1$$

Using partial fractions, we can also evaluate the integral

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) + C.$$

In fact, we have the following identities

$$\operatorname{artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right),$$
$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right),$$
$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right).$$

**4.2.4** Show that arsinh  $(x) = \ln (x + \sqrt{x^2 + 1})$ 

### 4.3 Further reading: The catenary problem



Figure 8: Profile of a heavy chain hanging under gravity.

One of the most famous problems where hyperbolic functions are used is in determining the profile of a heavy chain (of constant density  $\rho$ ) suspended from two points of equal height (known as a catenary curve).

To derive the differential equation satisfied by the profile y(x), we look at the forces acting on a small element of arc (inside the rectangular box in figure 8).



Figure 9: A small arc of heavy chain of length  $\delta s$ .

Let T(x) be the tensile force in the chain with constant horizontal component H (since the load has no x component) and vertical component V(x). In figure 9 the vertical components of the tensile force at either end of the arc are V and  $V + \delta V$ .

The mass of the arc will be  $\rho(\delta s)$ , so that the force due to gravity is  $\rho g(\delta s)$ .

The horizontal equilibrium is the trivial relation H = H, whereas the vertical equilibrium is the more informative

$$(V + \delta V) = V + \rho g(\delta s).$$

Dividing both sides by  $\delta x$  gives

$$\frac{\delta V}{\delta x} = \rho g \frac{\delta s}{\delta x}$$

From geometry, we also have the approximation

$$\frac{\delta y}{\delta x} \approx \frac{V}{H}.$$

We also have the approximation to the arclength  $\delta s$ 

$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2 \quad \Rightarrow \quad \frac{\delta s}{\delta x} \approx \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

Finally we take the limit  $\delta x \to 0$  so that  $\delta y \to 0$  and  $\delta s \to 0$  simultaneously. We then have the following equations

$$\frac{dV}{dx} = \rho g \frac{ds}{dx},$$

$$V = H \frac{dy}{dx},$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Putting these equations together gives the ODE satisfied by the profile y(x),

$$\frac{d^2y}{dx^2} = \frac{\rho g}{H} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Notes.

# 5 Linear second order nonhomogeneous ODEs, the Wronskian

By the end of this section, you should be able to answer the following questions:

- What is the form of the general solution to a linear, second order nonhomogeneous ODE?
- How do you define the Wronskian of an *n*th order linear second order homogeneous ODE?
- What is the relationship between the Wronskian and linearly independent solutions to a linear, second order homogeneous ODE?

ODEs can be split into two classes: linear and non-linear. Non-linear ODEs are generally very difficult to solve. Linear ODEs are simpler because their solutions have general properties which facilitate working with them. There are also well established methods for solving many linear ODEs of practical significance.

A second order ODE is called linear if it can be written in the form

$$y'' + p(x)y' + q(x)y = r(x).$$
(10)

Any second order ODE which cannot be written in this form is called non-linear. Note that y and its derivatives appear linearly and throughout we assume that p, q and r are continuous functions on some open interval I.

Over the next few sections we study linear second order ODEs. The motivation for studying second order ODEs is twofold. Firstly they have applications in mechanics and electric circuit theory, so anyone studying either of these fields will most likely come across second order ODEs. Secondly, the theory of linear second order ODEs is very similar to that of higher order linear ODEs, so that the transition to studying higher order linear ODEs would not require too many new ideas.

A great deal of this discussion can be found in the book *Elementary Differential Equations and Boundary Value Problems* by W.E. Boyce and R.C. DiPrima (ed. 10, Wiley, 2012).

### 5.1 Existence and uniqueness theorem

The key result for initial value problems related to second order linear ODEs is given as follows.

Consider the IVP

$$y'' + p(x)y' + q(x)y = r(x), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0,$$

where p, q and r are continuous functions on an open interval I that contains  $x_0$ . Then there is exactly one solution of the IVP which exists throughout I.

The proof of this theorem is beyond the scope of the course. Second order linear ODEs were introduced in MATH1052/MATH1072, and we first recall some important results and concepts, before looking more closely at certain aspects.

# 5.2 The superposition principle

Recall that if r(x) = 0 in equation (10), then we call the ODE homogeneous. If  $r(x) \neq 0$ , then the ODE is nonhomogeneous.

For any homogeneous linear equation, if  $y_1$  and  $y_2$  are solutions, so too is the linear combination  $Ay_1 + By_2$ . It is important to note that the superposition principle is not true for nonlinear equations and nonhomogeneous.

## 5.3 Linear independence

The functions  $f_1, f_2, \ldots, f_n$  are said to be *linearly dependent on an interval I* if there is a set of constants  $k_1, k_2, \ldots, k_n$ , not all zero, such that

$$k_1 f_1(x) + k_2 f_2(x) + \dots + k_n f_n(x) = 0$$

for all  $x \in I$ . The functions are said to be *linearly independent on* I if they are not linearly dependent there.

5.3.1 Example: linear independence of  $g_1(x) = 1$ ,  $g_2(x) = x$ ,  $g_3(x) = x^2$  on the interval  $-\infty < x < \infty$ 

# 5.4 General solution (homogeneous case), Wronskian of solutions

From the superposition principle, we can see that given two solutions,  $y_1$  and  $y_2$  of the ODE, we are able to construct an infinite family by taking the linear combination  $Ay_1 + By_2$ . The question that arises is whether or not this family contains all solutions.

A first step is to understand if we can choose constants A and B to satisfy given initial conditions. Let  $f_1$  and  $f_2$  be two differentiable functions on the interval I. The *Wronskian* of  $f_1$  and  $f_2$  at  $x \in I$  is defined as

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix} = f_1(x)f'_2(x) - f'_1(x)f_2(x).$$

Now let  $y_1$  and  $y_2$  be solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Is it possible to find A and B so that the initial conditions  $y(x_0) = y_0$  and  $y'(x_0) = y'_0$ are satisfied by  $y(x) = Ay_1(x) + By_2(x)$ ? We have the following results involving the Wronskian.

Suppose that  $y_1$  and  $y_2$  are two solutions of

y'' + p(x)y' + q(x)y = 0,

and consider the associated IVP with initial conditions

 $y(x_0) = y_0, y'(x_0) = y'_0.$ 

It is always possible to choose A and B so that

$$y = Ay_1(x) + By_2(x)$$

satisfies the IVP if and only if

 $W(y_1, y_2)(x_0) \neq 0.$ 

It is the next result that allows us to use the phrase general solution that was introduced in MATH1052/MATH1072. The proof makes use of the previous result.

Let  $y_1$  and  $y_2$  be two solutions of

$$y'' + p(x)y' + q(x)y = 0.$$

The family of solutions

$$y = Ay_1(x) + By_2(x)$$

with arbitrary coefficients A, B includes every solution of this ODE if and only if there is a value  $x_0$  such that  $W(y_1, y_2)(x_0) \neq 0$ . In such a case we call  $y_1$ and  $y_2$  a fundamental set of solutions.

# 5.5 Abel's Theorem

Let  $y_1$  and  $y_2$  be solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous on an open interval I. The Wronskian is given by

$$W(y_1, y_2)(x) = C e^{-\int p(x) \, dx},$$

where C is a constant that depends on  $y_1$  and  $y_2$ , but not x. Moreover  $W(y_1, y_2)(x)$  either is zero for all  $x \in I$ , or else is never zero in I.

# 5.6 Nonhomogenous linear ODEs

Now we consider equations of the form

$$y'' + p(x)y' + q(x)y = r(x), \quad r(x) \neq 0.$$
(11)

You should know from MATH1052/MATH1072 that the general solution on an open interval  ${\cal I}$  is of the form

$$y = y_H + y_P,$$

where  $y_H$  is the general solution of the homogeneous equation (with r(x) = 0) on I and  $y_P$  is a particular solution of (11) on I containing no arbitrary constants.

One strategy for finding the general solution is to first find  $y_H$ , then find  $y_P$ . See if you can work through the following example (review from MATH1052/MATH1072).

### **5.6.1** Example: $y'' - 2y' + y = e^x + x$

### 5.7 Further reading: The method of undetermined coefficients

In what follows, we determine a solution to the homogenous equation, and then *try* a form (with *undetermined coefficients*) for the particular solution which looks like it will result in the function on the right hand side.

The method of undetermined coefficients, as presented here, only works for the constant coefficient case:

$$y'' + ay' + by = r(x),$$

and r(x) contains exponentials, polynomials, sines and cosines, or sums and certain products of these functions.

Choose for  $y_P$  a form similar to r(x), involving unknown coefficients. The coefficients are then determined by substituting  $y_P$  into the ODE.

$r_i(x)$	$g_i(x)$	$r_i(x)$	$g_i(x)$
$ke^{\gamma x}$	$ae^{\gamma x}$	$k\cos\omega x,\\k\sin\omega x$	$a\cos\omega x + b\sin\omega x$
$\sum_{i=0}^{N} k_i x^i, \ N = 0, 1, 2, \dots$	$\sum_{i=0}^{N} a_i x^i$	$k e^{\alpha x} \cos \omega x,$ $k e^{\alpha x} \sin \omega x$	$e^{\alpha x}(a\cos\omega x + b\sin\omega x)$

We follow these basic steps.

- 1. Find a solution  $y_H$  to the corresponding homogeneous equation.
- 2. For  $r(x) = r_1(x) + r_2(x) + \ldots + r_n(x)$ , we first make a guess  $g(x) = g_1(x) + g_2(x) + \ldots + g_n(x)$  for  $y_P$ , where the  $g_i(x)$  correspond to the  $r_i(x)$  entries in the table above.
- 3. If a term  $g_i(x)$  appears in  $y_H$ , replace  $g_i(x)$  in the initial guess by  $xg_i(x)$ .
- 4. If any of the  $xg_i(x)$  from step 3 appear in  $y_H$ , replace  $xg_i(x)$  by  $x^2g_i(x)$ .
- 5. Substitute the modified guess g(x) into the left hand side of the ODE and equate coefficients on both sides. Once you have worked out the coefficients, the guess g(x) becomes  $y_P$ .

Notes.

# 6 Variation of parameters

By the end of this section, you should be able to answer the following questions:

- Under what conditions does the method work?
- What functions need to be determined first before using the method?
- How do you use the variation of parameters method to solve a nonhomogeneous linear second order ODE?

The method of undetermined coefficients is very easy to apply, but only works for constant coefficients with certain r(x). For other cases, the variation of parameters works well. The process is the following:

- Solve y'' + p(x)y' + q(x)y = 0, obtain a fundamental set of solutions  $y_1, y_2$  and calculate the Wronskian,  $W(y_1, y_2)(x) \equiv W$ .
- Set  $y_P = u(x)y_1(x) + v(x)y_2(x)$  and substitute into the ODE. We also impose the condition  $u'y_1 + v'y_2 = 0$ . We have the freedom to impose this extra arbitrary condition because we have two functions (u and v) and only one equation they need to satisfy arising from the ODE.
- We obtain

$$u(x) = -\int \frac{y_2 r}{W} dx, \quad v(x) = \int \frac{y_1 r}{W} dx.$$

This approach is a variant of the method of Reduction of Order, which prescribes that we take a solution, say  $y_1$  of the associated homogeneous equation and seek a particular solution of the form  $y_p = U(x)y_1$ . 6.1 Derive the formulae for u(x) and v(x) in the variation of parameters

6.2 Example:  $y'' - 4y' + 5y = 2e^{2x} / \sin x$ 



# 6.3 Repeat the previous example using reduction of order

# 6.4 Summary of ODE techniques and types of equations you should know

- First order, directly integrable
- First order, separable
- First order, linear, integrating factor
- First order existence and uniqueness criteria
- First order, exact
- Second order homogeneous, linear, constant coefficients
- Second order nonhomogeneous, constant coefficients, method of undetermined coefficients for certain cases
- Second order nonhomogeneous, variation of parameters.
- Reduction of order

Notes.

# 7 Vector spaces

By the end of this section, you should be able to understand:

- The definition of a vector space.
- Many new examples of vector spaces.
- The significance of a basis.
- How to find the transition matrix from one basis to another.

**Notation.** For the next set of lectures on linear algebra,  $\mathbb{F}$  stands for  $\mathbb{R}$  or  $\mathbb{C}$ . Thus, if a statement holds for or applies to both number sets, we may simply state it for  $\mathbb{F}$ . Elements of  $\mathbb{F}$  are often called *scalars*.

### 7.1 Definition

Let V be a nonempty set on which are defined operations "+" (called addition) and " $\cdot$ " (called scalar multiplication). V is a **vector space** (over  $\mathbb{F}$ ) if the following hold for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and all  $k, \ell \in \mathbb{F}$ :

(V1)  $\mathbf{u} + \mathbf{v} \in V$  (closure)

(V2)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  (additive commutativity)

(V3)  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$  (additive associativity)

(V4)  $\exists 0 \in V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  (zero vector, or additive identity)

(V5) For each  $\mathbf{u} \in V$ ,  $\exists (-\mathbf{u}) \in V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$  (additive inverse)

$$(\mathbf{V6}) \quad k \cdot \mathbf{u} \in V$$

(V7)  $k \cdot (\mathbf{u} + \mathbf{v}) = k \cdot \mathbf{u} + k \cdot \mathbf{v}$  (multiplicative-additive distributivity)

(V8)  $(k + \ell) \cdot \mathbf{u} = k \cdot \mathbf{u} + \ell \cdot \mathbf{u}$  (additive-multiplicative distributivity)

(V9)  $k \cdot (\ell \cdot \mathbf{u}) = (k\ell) \cdot \mathbf{u}$  (multiplicative-multiplicative distributivity)

(V10) 1 · u = u (multiplicative identity)

The scalar multiplication symbol is often omitted. Elements of a vector space are usually called vectors.

7.2 Example:  $\mathbb{F}^n$  – set of *n*-tuples

7.3 Example:  $M_{m,n}(\mathbb{F})$  – set of  $m \times n$  matrices

7.4 Example: C[a, b] – set of continuous real-valued functions on [a, b]

7.5 Example:  $P_n(\mathbb{F})$  – set of polynomials of degree at most n

7.6 Example: Set of solutions to a homogeneous linear ODE

### 7.7 Familiar concepts in linear algebra

Here we lists several concepts with which you should already be familiar.

#### • Linear combination

For  $v_1, v_2, \ldots, v_n \in V$  and  $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ , we call

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

a linear combination of the vectors  $v_1, v_2, \ldots, v_n$ .

#### • Linear independence

A non-empty set of vectors  $S = \{v_1, v_2, \ldots, v_n\}$  in V is said to be *linearly dependent* if there exist scalars  $\alpha_1, \alpha_2, \ldots, \alpha_n$  not all zero such that

 $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0.$ 

Otherwise, S is called *linearly independent*, i.e. S is linearly independent if

 $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$ 

#### • Subspace

A subset  $W \subseteq V$  is called a *subspace* if W is also a real vector space with the same addition and scalar multiplication. In particular, W is required to close under addition and scalar multiplication.

#### • Span

The span of a non-empty set of vectors  $S = \{v_1, v_2, \ldots, v_n\}$  in V is the set of all linear combinations of vectors in S, denoted span(S). The set span(S) is a subspace of V.

#### 7.8 Basis

Let  $\beta = {\mathbf{v}_1, \dots, \mathbf{v}_n}$  be a set of vectors in the vector space V.  $\beta$  is a **basis** for V if

- (B1)  $\beta$  is linearly independent;
- (B2)  $\beta$  spans V.

Note that the notion of a basis is only defined here for finite sets. A nonzero vector space is **finite-dimensional** if it contains a finite set of vectors that forms a basis. If no such set exists, the vector space is **infinite-dimensional**.

Let V be a finite-dimensional vector space. The number of vectors in any basis for V is the same, and this number is known as the **dimension** of V.

An **ordered basis** for a vector space is a basis endowed with a specific order. For some vector spaces, there is a canonical ordered basis, called a **standard basis**.

# 7.9 Decomposition theorem

Let  $\beta = {\mathbf{v}_1, \ldots, \mathbf{v}_n}$  be a set of vectors in the vector space V. Then,  $\beta$  is a basis for V iff each  $\mathbf{w} \in V$  can be uniquely expressed as a linear combination of vectors in  $\beta$ .



### 7.10 Transition matrix

Let  $\beta = {\mathbf{v}_1, \ldots, \mathbf{v}_n}$  be an ordered basis for the vector space V. For  $\mathbf{u} \in V$ , let  $a_1, \ldots, a_n$  be (the unique) scalars such that

$$\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i.$$

The coordinate vector of **u** relative to  $\beta$  is given by

$$[\mathbf{u}]_{\beta} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Another common notation for this is  $[\mathbf{u}]^{\beta}$ .

Let  $\beta'$  be another ordered basis for V. The coordinate vector of **u** relative to  $\beta'$  is thus denoted by  $[\mathbf{u}]_{\beta'}$ . The **transition matrix** from  $\beta$  to  $\beta'$ , denoted by  $P_{\beta \to \beta'}$ , relates the two coordinate vectors of **u** as

$$[\mathbf{u}]_{\beta'} = P_{\beta \to \beta'} [\mathbf{u}]_{\beta}.$$

If  $\beta''$  is yet another ordered basis for V, then

$$P_{\beta' \to \beta''} P_{\beta \to \beta'} = P_{\beta \to \beta''} \implies \qquad \Longrightarrow \qquad P_{\beta' \to \beta} P_{\beta \to \beta'} = P_{\beta \to \beta} = I,$$

where I is the  $n \times n$  identity matrix.

To illustrate, let us consider the two ordered bases  $\beta = \{1, x\}$  and  $\beta' = \{1 + x, 2x\}$ for  $P_1(\mathbb{F})$ . As the vector (or polynomial)  $\mathbf{u} = a + bx$  also can be written as

$$\mathbf{u} = a(1+x) + \frac{1}{2}(b-a)(2x),$$

we have

$$[\mathbf{u}]_{\beta} = \begin{pmatrix} a \\ b \end{pmatrix}$$
 and  $[\mathbf{u}]_{\beta'} = \begin{pmatrix} a \\ \frac{1}{2}(b-a) \end{pmatrix}$ 

The corresponding transition matrix  $P_{\beta \to \beta'}$  is given as follows:



Notes.

# 8 Real inner product spaces

By the end of this section, you should understand the definition of inner product space, and be aware of many new examples.

Every vector space in this section is real.

#### 8.1 Dot product

The familiar **dot product**,  $\mathbf{u} \cdot \mathbf{v}$ , of the two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$  is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

where  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$ . This is readily generalised to  $\mathbb{R}^n$  by

 $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n$ , where  $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n).$ 

This dot product has the following key properties:

#### 8.2 Inner product

Inspired by the dot product on  $\mathbb{R}^n$ , we define a so-called inner product on a general real vector space by elevating the key properties of the dot product to axioms.

Accordingly, an **inner product** on V is a function that takes each ordered pair  $(\mathbf{u}, \mathbf{v})$  of elements of V to a real number, denoted by  $\langle \mathbf{u}, \mathbf{v} \rangle$ , such that for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and all  $k \in \mathbb{R}$ :

(I1) 
$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

(I2) 
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

(I3) 
$$\langle k \cdot \mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$$

 $(\mathbf{I4}) \quad \langle \mathbf{u}, \mathbf{u} \rangle \ge 0$ 

(I5)  $\langle \mathbf{u}, \mathbf{u} \rangle = 0$  iff  $\mathbf{u} = \mathbf{0}$  (where **0** is the unique zero vector)

A vector space with an inner product associated to it is called an **inner product space**. As we are assuming that all vector spaces are real in this section, we have thus introduced the notion of a *real* inner product space. One can also define inner products on complex vector spaces, thereby introducing *complex* inner product spaces, but they are beyond the scope of these lectures.

# 8.3 Example: Weighted dot product

# 8.4 Example: Inner product generated by a matrix

# 8.5 Example: Inner product on $M_{n,n}(\mathbb{R})$

8.6 Example: Standard inner product on  $P_n(\mathbb{R})$ 

- 8.7 Example: Evaluation inner product on  $P_n(\mathbb{R})$

# 8.8 Example: Inner product on C[a, b]

Notes.

#### Solving linear equations 9

By the end of this section, you should be able to answer the following questions:

- How do you use Gaussian elimination to find a row echelon form of a matrix?
- What are the three cases for solutions to systems of linear equations?
- How do you solve a system of linear equations?
- What is the rank of a matrix?

Say we have m linear equations in n variables:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
  

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
  

$$\vdots$$
  

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

We can write these equations in matrix form:  $A\mathbf{x} = \mathbf{b}$ .

 $A = [a_{ij}]$  is the  $m \times n$  coefficient matrix.  $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  is the column vector of unknowns, and  $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$  is the column vector of the right hand side

Note:  $a_{ij}, b_j \in \mathbb{R}$  or  $\mathbb{C}$ .

#### 9.1 Gaussian Elimination

To solve  $A\boldsymbol{x} = \boldsymbol{b}$ :

write augmented matrix:  $[A|\mathbf{b}]$ .

**1.** Find the left-most non-zero column, say column *j*.

**2.** Interchange top row with another row if necessary, so top element of column i is non-zero. (The **pivot**.)

**3.** Subtract multiples of row 1 from all other rows so all entries in column j below the top are then 0.

4. Cover top row; repeat 1 above on rest of rows.

Continue until all rows are covered, or until only 00...0 rows remain.

The result is a triangular system, easily solved by *back substitution*: solve the last equation first, then 2nd last equation and so on.
## 9.1.1 Example

Use Gaussian elimination to solve:

$$x_3 - x_4 = 2$$
  
-9x<sub>1</sub> - 2x<sub>2</sub> + 6x<sub>3</sub> - 12x<sub>4</sub> = -7  
3x<sub>1</sub> + x<sub>2</sub> - 2x<sub>3</sub> + 4x<sub>4</sub> = 2  
2x<sub>3</sub> = 6

#### 9.1.2 Definition (row echelon form)

A matrix is in *row echelon form* (r.e.f.) if each row after the first starts with *more* zeros than the previous row (or else rows at bottom of matrix are all zeros).

The Gauss algorithm converts any matrix to one in row echelon form. The 2 matrices are *equivalent*, that is, they have the same solution set.

#### 9.1.3 Elementary row operations

**1.**  $r_i \leftrightarrow r_j$ : swap rows *i* and *j*. **2.**  $r_i \rightarrow r_i - cr_j$ : replace row *i* with

(row i minus c times row j).

**3.**  $r_i \rightarrow cr_i$ :

replace row i with c times row i, where  $c \neq 0$ .

The Gauss algorithm uses only 1 and 2.

#### 9.2 Possible solutions for Ax = b

Consider the r.e.f. of  $[A|\mathbf{b}]$ . Then we have three possibilities:

(1) *Exactly one* solution; here the r.e.f. gives each variable a single value, so the number of variables, n, equals the number of non-zero rows in the r.e.f.

(2) Infinitely many solutions; here the number of non-zero rows of the r.e.f. is less than the number of variables.

For cases (1) and (2), the system is said to be *consistent*.

(3) No solution; when one row of r.e.f. is  $(0 \ 0 \ \dots \ d)$  with  $d \neq 0$ . We can't solve  $0x_1 + 0x_2 + \dots + 0x_m = d$  if  $d \neq 0$ ; it says 0 = d. In this case the system is said to be *inconsistent*.

Note that a homogeneous system has  $\boldsymbol{b} = \boldsymbol{0}$ , i.e., all zero RHS. Then we always have at least the trivial solution,  $x_i = 0, 1 \leq i \leq n$ .

Let A be an  $m \times n$  matrix:

- The subspace of  $\mathbb{R}^n$  spanned by the rows of A is called the *row space* of A, denoted  $\operatorname{Row}(A)$ .
- The subspace of  $\mathbb{R}^m$  spanned by the columns of A is called the *column space* of A, denoted  $\operatorname{Col}(A)$ .
- The subspace of  $\mathbb{R}^n$  that is the solution space of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$  is called the *nullspace* of A (denoted N(A)), and its dimension is often referred to as the *nullity*, denoted nullity(A).
- The rank of A, denoted  $\operatorname{rank}(A)$ , is the dimension of the column space of A.

## 9.2.1 Examples

$$\begin{array}{rcrcrcrcr}
x_1 + x_2 - & x_3 &= & 0\\ 
2x_1 - x_2 &= & 0\\ 
4x_1 + x_2 - 2x_3 &= & 1
\end{array}$$

These examples highlight some important results for linear systems.

A system of linear equations  $A\boldsymbol{x} = \boldsymbol{b}$  is consistent if and only if  $\boldsymbol{b}$  is in the column space of A.

If  $\boldsymbol{x}_0$  is any solution of a consistent linear system  $A\boldsymbol{x} = \boldsymbol{b}$ , and if  $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$  is a basis for the nullspace of A, then every solution of  $A\boldsymbol{x} = \boldsymbol{b}$  can be expressed in the form

$$\boldsymbol{x} = \boldsymbol{x}_0 + c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_k \boldsymbol{v}_k.$$

In other words, if  $A\mathbf{x} = \mathbf{b}$  corresponds to a consistent linear system of m equations in n unknowns (so A is understood to be  $m \times n$ ), and if rank(A) = r, then the solution contains n - r free parameters.

Notes.

### 10 Orthogonality

The goal of this section is to develop our understanding of orthogonality in the context of inner product spaces.

#### 10.1 Norm

The norm (or magnitude or length) of an element  $\mathbf{v} = (v_1, \ldots, v_n)$  of  $\mathbb{R}^n$  is given by the familiar expression

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

There is a similar notion for any real inner product space V. The **norm** of a vector  $\mathbf{v} \in V$ , denoted by  $||\mathbf{v}||$ , is thus defined by

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

A vector with norm 1 is called a **unit vector**.

How would we define the *distance*,  $d(\mathbf{u}, \mathbf{v})$ , between two vectors  $\mathbf{u}, \mathbf{v} \in V$ ? A natural notion of distance between two vectors should be independent of the order we happen to be viewing them. That is, we want the distance measure to be symmetric:  $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ . Again using  $\mathbb{R}^n$  as inspiration, we now define the **distance** between two vectors  $\mathbf{u}, \mathbf{v} \in V$  as

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

The symmetry  $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$  follows from

$$\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v} - \mathbf{u}, \mathbf{v} - \mathbf{u} \rangle.$$

Note that the notions of norm and distance are relative to the inner product used! For example, with the inner product given in Example 8.8, the norm of a real-valued continuous function on [-1, 1] is given as follows:

As in  $\mathbb{R}^n$  with inner product given by the usual dot product, we say that two vectors  $\mathbf{u}, \mathbf{v} \in V$  are **orthogonal** if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

We need a bit of preparation before we can talk more generally about the *angle* between two vectors, see Section 10.7.

## 10.2 Pythagorean theorem

Let V be a real inner product space, and let  $\mathbf{u}, \mathbf{v} \in V$ . Then,

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

### 10.3 Cauchy-Schwarz inequality

Let V be a real inner product space, and let  $\mathbf{u}, \mathbf{v} \in V$ . Then,

 $|\langle \mathbf{u}, \mathbf{v} \rangle| \le ||\mathbf{u}|| \, ||\mathbf{v}||.$ 

Moreover, this inequality is an equality if and only if  ${\bf u}$  or  ${\bf v}$  is a scalar multiple of the other vector.

# 10.4 Triangle inequality

Let V be a real inner product space, and let  $\mathbf{u}, \mathbf{v} \in V$ . Then,

 $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$ 

**10.5 Example:** C[-1, 1]

**10.6 Example:**  $P_2(\mathbb{R})$ 

# 10.7 Angle between two vectors

### 10.8 Orthogonal complement

Let U be a subset of the real inner product space V. The **orthogonal complement** of U, denoted by  $U^{\perp}$ , is the set of all vectors in V that are orthogonal to every vector in U. That is,

$$U^{\perp} = \{ \mathbf{v} \in V \, | \, \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for every } \mathbf{u} \in U \}.$$

This is a vector space with addition and scalar multiplication inherited from V.

# 10.8.1 For $A \in M_{m,n}(\mathbb{R})$ , $\operatorname{Row}(A)^{\perp} = N(A)$ with respect to the Euclidean inner product



In fact,  $U^{\perp}$  is an example of a *subspace*. Indeed, a nonempty subset W of a vector space V is a **subspace** of V if it is a vector space under the addition and scalar multiplication defined on V. To verify that a subset is a subspace, one checks the following:

Notes.

# 11 Gram-Schmidt process

The goal of this section is to discuss orthogonal projections, and develop a way of constructing an orthonormal basis for an inner product space.

### 11.1 Orthogonal set

Let V be a real inner product space. A nonempty set of vectors in V is **orthogonal** if each vector in the set is orthogonal to all the other vectors in the set. That is, the set  $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V$  is orthogonal if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \qquad i \neq j.$$

Let S be a finite set of vectors in V such that  $\mathbf{0} \notin S$ . Then,

S orthogonal  $\implies$  S linearly independent.

### 11.2 Orthonormal basis

An orthogonal set of vectors in V is called **orthonormal** if all the vectors in the set are unit vectors. That is, the set  $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset V$  is orthonormal if

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{i,j},$$

where the **Kronecker delta** is defined by

$$\delta_{i,j} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Examples in  $\mathbb{R}^n$  endowed with the usual dot product are given as follows:

An **orthonormal basis** for V is a basis for V that is also an orthonormal set.

# 11.3 Decomposition theorem

Let  $S = {\mathbf{e}_1, \dots, \mathbf{e}_n}$  be an orthonormal basis for V, and let  $\mathbf{u} \in V$ . Then,

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \ldots + \langle \mathbf{u}, \mathbf{e}_n \rangle \mathbf{e}_n$$

and

$$||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{e}_1 \rangle^2 + \ldots + \langle \mathbf{u}, \mathbf{e}_n \rangle^2.$$

#### 11.4 Orthogonal projection

Let U be a finite-dimensional subspace of the real inner product space V. Then, each  $\mathbf{v} \in V$  can be written in a unique way as

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \qquad \mathbf{u} \in U, \quad \mathbf{w} \in U^{\perp}.$$

In the proof, we will assume that U has an orthonormal basis  $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ . As we will see in Section 11.6, this assumption is redundant since every finite-dimensional inner product space has such a basis.

The vector  $\mathbf{u} \in U$  is called the **orthogonal projection of v onto** U and is given by

$$\operatorname{Proj}_U(\mathbf{v}) = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \ldots + \langle \mathbf{v}, \mathbf{e}_k \rangle \mathbf{e}_k.$$

Likewise, the vector  $\mathbf{w} \in U^{\perp}$  is called the **orthogonal projection of v onto**  $U^{\perp}$  and is given by

$$\operatorname{Proj}_{U^{\perp}}(\mathbf{v}) = \mathbf{v} - \operatorname{Proj}_{U}(\mathbf{v}).$$

One can show that

$$\dim V = \dim U + \dim U^{\perp}.$$

This can be very helpful when determining the orthogonal complement of a subspace U. Indeed, suppose you have managed to find dim  $V - \dim U$  linearly independent vectors that are all orthogonal to U. Then, these vectors will, in fact, form a basis for  $U^{\perp}$ . This property will be used in the next example.

# 11.5 Example: Orthogonal projection in $\mathbb{R}^3$

Let  $\mathbb{R}^3$  be endowed with the usual dot product, and let

$$U = \operatorname{span}(\{(0, 1, 0), (-\frac{4}{5}, 0, \frac{3}{5})\}), \qquad \mathbf{v} = (1, 1, 1).$$

Find the orthogonal projections of  $\mathbf{v}$  onto U and  $U^{\perp}$ .

### 11.6 Construction of orthonormal basis

It is often convenient to have an orthonormal basis for a given finite-dimensional inner product space. The following algorithm turns a linearly independent set of vectors into an orthonormal set of vectors with the same span as the original set. Applying the algorithm to a basis thus turns the basis into an orthonormal basis. Hence:

Every finite-dimensional real inner product space has an orthonormal basis.

Let  $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$  be a linearly independent set of vectors in the real inner product space V. The corresponding **Gram-Schmidt process** is the following algorithm:

# 11.7 Example: Orthonormal basis for $P_1(\mathbb{R})$

Notes.

## 12 Least squares approximation

### 12.1 Least squares problem - minimising distance to a subspace

A recurring problem in linear algebra, and in its myriad of applications, is the following:

• Given a vector **v** in a real inner product space V, give the best approximation to **v** in a finite-dimensional subspace U of V.

Question: What do we mean by "best approximation"?

Answer: Seek  $\mathbf{u} \in U$  that minimises  $||\mathbf{v} - \mathbf{u}||$ . Equivalently, find a vector in a subspace (for example, corresponding to a point on a plane in  $\mathbb{R}^3$ ), of minimal distance to a given vector in the ambient vector space (in this example, corresponding to a point in  $\mathbb{R}^3$ ). Concretely, let  $\mathbf{v} \in V$ . Then, the problem is to

find  $\mathbf{u} \in U$  such that  $d(\mathbf{u}, \mathbf{v})$  is as small as possible.

This problem is called the "least squares problem."

**Theorem** (Best Approximation Theorem). If U is a finite-dimensional subspace of a real inner product space V, and if  $\mathbf{v} \in V$ , then  $\operatorname{Proj}_U(\mathbf{v})$  is the best approximation to  $\mathbf{v}$  from U in the sense that

$$||\mathbf{v} - \operatorname{Proj}_U(\mathbf{v})|| < ||\mathbf{v} - \mathbf{u}|| \quad \forall \mathbf{u} \in U : \mathbf{u} \neq \operatorname{Proj}_U(\mathbf{v}).$$

In practice, rather than work with minimising  $||\mathbf{v}-\mathbf{u}||$ , we minimise  $||\mathbf{v}-\mathbf{u}||^2$  (same outcome, avoid square root). Then Best Approximation Theorem  $\implies \operatorname{Proj}_U(\mathbf{v})$  is the best approximation

 $\iff$   $\mathbf{u} = \operatorname{Proj}_U(\mathbf{v})$  is the vector that minimises  $||\mathbf{v} - \mathbf{u}||^2$ .

How to find  $\operatorname{Proj}_{U}(\mathbf{v})$ ?

Solution 1. Use Gram-Schmidt process to construct an orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$  of U. Then

$$\operatorname{Proj}_U(\mathbf{v}) = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \ldots + \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{e}_n.$$

#### Solution 2.

Let  $\gamma = {\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n}$  be a basis of the subspace U. We seek coefficients  $\alpha_1, \alpha_2, \ldots, \alpha_n$  that minimise  $||\mathbf{v} - (\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n)||^2$ .

$$\begin{aligned} ||\mathbf{v} - (\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_n \mathbf{u}_n)||^2 \\ = \langle \mathbf{v} - (\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_n \mathbf{u}_n), \mathbf{v} - (\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_n \mathbf{u}_n) \rangle \\ = \langle \mathbf{v}, \mathbf{v} \rangle - 2\alpha_1 \langle \mathbf{v}, \mathbf{u}_1 \rangle - 2\alpha_2 \langle \mathbf{v}, \mathbf{u}_2 \rangle - \dots - 2\alpha_n \langle \mathbf{v}, \mathbf{u}_n \rangle + \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\ = E(\alpha_1, \alpha_2, \dots, \alpha_n). \end{aligned}$$

E is the "error (real valued function)."

From MATH1052/MATH1072: we minimise E. Set  $\nabla E = \mathbf{0}$ , which gives

$$\frac{\partial E}{\partial \alpha_k} = -2 \langle \mathbf{v}, \mathbf{u}_k \rangle + 2 \sum_{l=1}^n \alpha_l \langle \mathbf{u}_k, \mathbf{u}_l \rangle = 0.$$

This is n equations in n unknowns, which may be expressed in matrix form;

$$\begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle & \langle \mathbf{u}_1, \mathbf{u}_2 \rangle \dots \langle \mathbf{u}_1, \mathbf{u}_n \rangle \\ \langle \mathbf{u}_2, \mathbf{u}_1 \rangle & \langle \mathbf{u}_2, \mathbf{u}_2 \rangle \dots \langle \mathbf{u}_2, \mathbf{u}_n \rangle \\ \vdots & \vdots & \\ \langle \mathbf{u}_n, \mathbf{u}_1 \rangle & \langle \mathbf{u}_n, \mathbf{u}_2 \rangle \dots \langle \mathbf{u}_n, \mathbf{u}_n \rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \langle \mathbf{v}, \mathbf{u}_1 \rangle \\ \langle \mathbf{v}, \mathbf{u}_2 \rangle \\ \vdots \\ \langle \mathbf{v}, \mathbf{u}_n \rangle \end{pmatrix}$$

*Note:* If  $\gamma$  is orthonormal  $\implies$  matrix on LHS= *I* (the identity matrix)  $\implies \alpha_i = \langle \mathbf{v}, \mathbf{u}_i \rangle$ , which coincides with Solution 1, as expected.

#### **12.2** Inconsistent linear systems

In applications, one is often faced with over-determined linear systems. For example, we may have a bunch of data points that we have reasons to believe should fit on a straight line. But real-life data points rarely match predictions exactly. The goal in this section is to develop a method for obtaining the best fit or approximation of a specified kind to a given set of data points.

To this end, let  $A \in M_{m,n}(\mathbb{R})$  and  $\mathbf{b} \in M_{m,1}(\mathbb{R})$ . For m > n, the linear system described by  $A\mathbf{x} = \mathbf{b}$  is over-determined and does not in general have a solution. However, we may be satisfied if we can find  $\hat{\mathbf{x}} \in M_{n,1}(\mathbb{R})$  such that  $A\hat{\mathbf{x}}$  is as 'close' to **b** as possible. That is, we seek to minimise  $||\mathbf{b} - A\mathbf{x}||$  for given A and **b**. For computational reasons, one usually minimises  $||\mathbf{b} - A\mathbf{x}||^2$  instead. A solution  $\hat{\mathbf{x}}$  to this minimisation problem is referred to as a **least squares solution**.

First, let us recall the **column space** of the matrix A as

$$\operatorname{Col}(A) = \operatorname{span}(\{\mathbf{a}_1, \dots, \mathbf{a}_n\}),$$

where  $\mathbf{a}_1, \ldots, \mathbf{a}_n \in M_{m,1}(\mathbb{R})$  are the columns of A. Since

$$\mathbf{a}_1 = A \begin{pmatrix} 1\\0\\\vdots\\0 \end{pmatrix}, \quad \mathbf{a}_2 = A \begin{pmatrix} 0\\1\\\vdots\\0 \end{pmatrix}, \dots, \quad \mathbf{a}_n = A \begin{pmatrix} 0\\0\\\vdots\\1 \end{pmatrix},$$

we have

$$\operatorname{Col}(A) = \operatorname{span}\left(\left\{A\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, A\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots, A\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\right\}\right) = \left\{A\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} \mid x_1, x_2, \dots, x_n \in \mathbb{R}\right\}$$
$$= \left\{A\mathbf{x} \mid \mathbf{x} \in M_{n,1}(\mathbb{R})\right\}.$$

It follows that  $A\hat{\mathbf{x}} \in \text{Col}(A)$ . We are thus in the situation described in Section 12.1.

Consequently, we know that

$$A\hat{\mathbf{x}} = \operatorname{Proj}_{\operatorname{Col}(A)}(\mathbf{b}),$$

but we are still faced with the task of disentangling  $\hat{\mathbf{x}}$  from  $A\hat{\mathbf{x}}$ . Note that, although  $A\hat{\mathbf{x}} = \operatorname{Proj}_{\operatorname{Col}(A)}(\mathbf{b})$  is *uniquely* given in terms of A and **b**, the ensuing result for  $\hat{\mathbf{x}}$  need *not* be.

**Solution 3.** For the current special application, i.e. least squares solutions of linear systems, we have a more direct (and simpler) method. In the following we describe this method and give some examples.

# **12.3** $Col(A)^{\perp} = N(A^T)$

The orthogonal complement of the column space of A is the null space of  $A^T$ .

#### 12.4 Solving for $\hat{x}$

Recall from Section 11.4 that  $\mathbf{v} - \operatorname{Proj}_U(\mathbf{v}) = \operatorname{Proj}_{U^{\perp}}(\mathbf{v}) \in U^{\perp}$  for any finite-dimensional subspace U of V and  $\mathbf{v} \in V$ . Since  $A\hat{\mathbf{x}} = \operatorname{Proj}_{\operatorname{Col}(A)}(\mathbf{b})$ , we thus have

 $\mathbf{b} - A\hat{\mathbf{x}} \in \mathrm{Col}(A)^{\perp}.$ 

Because  $\operatorname{Col}(A)^{\perp} = N(A^T)$ , it follows that  $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$ , so

 $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}.$ 

Since  $A \in M_{m,n}(\mathbb{R}) \Rightarrow A^T A \in M_{n,n}(\mathbb{R})$ , the equation  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$  can always be solved for  $\hat{\mathbf{x}}$ , for example by Gaussian elimination. However, the solution need not be unique. Indeed, the solution is unique if and only if  $A^T A$  is invertible, in which case

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}.$$

A key to determining whether  $A^{T}A$  is invertible is the following result:

 $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$  is linearly independent  $\iff A^T A$  is invertible.

But how does that help? Well, recall that m > n so that A may be describing an over-determined linear system. In fact, in case the linear system encodes experimental or empirical data, m is likely to be much larger than n. The columns of A are then very likely to be linearly independent, and  $A^T A$  would indeed be invertible.

### 12.5 Fitting a curve to data

Experiments yield data (assume  $x_i$  distinct and exact)

 $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ 

which include measurement error. Theory may predict polynomial relation between x and y. We seek a least squares polynomial function of best fit (e.g. least squares line of best fit or regression line).

**Example: Quadratic fit.** Suppose some physical system is modeled by a quadratic function p(t). Data in the form (t, p(t)) have been recorded as

(1,5), (2,2), (4,7), (5,10).

Find the least squares approximation for p(t).

Notes.

### 13 Least squares function approximation

Given a function  $f \in C[a, b]$ , find the best approximation to f using only functions from a specified subspace U of C[a, b].

Interpret "best possible" in the sense of least squares.

Consider g as an approximation to f.

At point  $x_0$  the error is  $|f(x_0) - g(x_0)|$ . For the entire interval, define error as  $\int_a^b |f(x) - g(x)| dx$ .

This is area between curves.

An easier definition (and one more amenable to calculations) is the *mean squared* error (MSE)

$$MSE = \int_{a}^{b} (f(x) - g(x))^2 dx.$$

Recall the integral inner product on C[a, b];

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{a}^{b} p(x) q(x) \ dx$$

$$\implies \text{MSE} = ||\mathbf{f} - \mathbf{g}||^2 = \langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle = \int_a^b (f(x) - g(x))^2 \, dx.$$

# 13.1 Example: sin(x)

Find the least squares approximation for  $\sin x$  in the subspace of  $C[0, \pi]$  spanned by  $\{1, x, x^2\}$ . Use the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^\pi p(x) q(x) \ dx.$$

# 13.2 Fourier coefficients

In  $C[0, 2\pi]$ , the set

$$\beta_n = \left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \cos kx \, | \, k = 1, \dots, n \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin kx \, | \, k = 1, \dots, n \right\},$$

where  $n \in \mathbb{N}_0$ , is orthonormal with respect to the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^{2\pi} f(x) g(x) dx.$$

It follows that  $\beta_n$  is an orthonormal basis for the (2n + 1)-dimensional subspace  $W_n = \operatorname{span}(\beta_n)$  of  $C[0, 2\pi]$ . The orthogonal projection of  $\mathbf{f} \in C[0, 2\pi]$  onto  $W_n$  is given by  $\operatorname{Proj}_{W_n}(\mathbf{f})$ . In the limit  $n \to \infty$ , the corresponding approximation of f(x) yields the **Fourier series** of f(x) over the interval  $[0, 2\pi]$ :

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx, \qquad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx,$$

are the associated Fourier coefficients.

### 14 Understanding determinants

By the end of this section, you should be able to answer the following questions:

- What is the motivation for the definition of the determinant of a  $2 \times 2$  matrix?
- What does the  $2 \times 2$  determinant tell us about linear transformations in the plane?

Recall from MATH1051/MATH1071 that a  $2 \times 2$  matrix can be viewed as a transformation in the plane. For example, the equation

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

can be intepreted as in the following diagram.



A matrix can be viewed as either transforming vectors or *points* in the plane. We also consider what happens to a set of points under such a transformation. For example, a curve, or a region in the plane.

It is also interesting to consider general features of linear transformations. There are two fundamental properties of linear transformations:

- 1. The origin remains fixed under a linear transformation.
- 2. Straight lines map to straight lines under a linear transformation.

Given the second property, we may consider how to quantify the amount by which a linear transformation stretches or contracts a region in the plane by first consider the image of a unit square.

### 14.1 The effect of a linear transformation on a grid

We start with a straightforward example: Describe the effect of the linear transformation  $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$  on the unit square  $S = \{(x, y) \mid 0 \le x, y \le 1\}$ .

Now consider a grid of lines parallel to the x and y axes.



These grid lines remain parallel and evenly spaced under a linear transformation.

### 14.2 The determinant

It follows that the effect of a linear transformation on any grid square, regardless of the size, must be the same. It is then helpful to approximate any region in the x-y plane by arbitrarily small grid squares and arrive at the same conclusion.

Under a linear transformation A, the area of any region in the x-y plane scales by the same amount. This amount (up to a sign) is called the *determinant of* A, denoted |A| or det(A). In the case

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \det(A) = ad - bc.$$
#### 14.3 Sign of the determinant and orientation

For a linear transformation in the plane, A, if det(A) < 0, this implies that the region has undergone a "flip" or change in orientation. This is made clear in the following example.

Example: Compare the effect of  $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$  on the unit square  $\{(x, y) \mid 0 \le x, y \le 1\}.$ 

### 14.4 Further examples

Consider the effect on the unit square under the following linear transformations.

1. 
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
  
2. 
$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$
  
3. 
$$\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$
  
4. 
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Notes.

# 15 Eigenvalues and eigenvectors

By the end of this section, you should be able to answer the following questions:

- How do you find the eigenvalues and eigenvectors of a given square matrix?
- What are some simple properties of eigenvalues and eigenvectors?
- Prove that the eigenvectors corresponding to distinct eigenvalues are linearly independent.

A great deal of this section should be familiar to you. We start by recalling some results on vector spaces associated with matrices.

#### 15.1 Non-singular matrices

For  $n \times n$  square matrix A, we have several conditions for the existence of  $A^{-1}$ . The the following statements are equivalent:

- **1.** A is non-singular.
- **2.**  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution  $\mathbf{x} = \mathbf{0}$ .
- **3.** If U is a r.e.f. for A, then U has no row of all zeros.
- **4.**  $A\mathbf{x} = \mathbf{b}$  has a solution for every *n*-dimensional column vector  $\mathbf{b}$ .
- **5.** det(A)  $\neq 0$ .
- 6. The columns of A are linearly independent.
- **7.** The rows of A are linearly independent.
- 8. nullity(A) = 0.
- **9.** rank(A) = n.

#### 15.2 Eigenvalues and eigenvectors

Let A be a square matrix. Then an *eigenvector* of A is a vector  $\mathbf{v} \neq \mathbf{0}$  such that

$$A\boldsymbol{v} = \lambda \boldsymbol{v},$$

for some scalar  $\lambda$ . We call  $\lambda$  the *eigenvalue* corresponding to  $\boldsymbol{v}$ . If  $\boldsymbol{v}$  is an eigenvector of A, then so is  $t\boldsymbol{v}$  for any scalar  $t \neq 0$ .

We have

$$A \boldsymbol{v} = \lambda \boldsymbol{v} = \lambda I \boldsymbol{v} \quad \Rightarrow \quad (A - \lambda I) \boldsymbol{v} = \boldsymbol{0}.$$

Hence  $\boldsymbol{x} = \boldsymbol{v}$  is a non-trivial solution to the homogeneous system of equations  $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ , and conversely, if there is a non-trivial solution then  $\lambda$  is an eigenvalue of A. Thus:

 $\lambda$  is an eigenvalue of  $A \Leftrightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$  has a non-trivial solution  $\Leftrightarrow A - \lambda I$  is singular  $\Leftrightarrow \det(A - \lambda I) = 0.$ 

For an  $n \times n$  matrix A, det $(A - \lambda I)$  is a polynomial of degree n in  $\lambda$ , called the *characteristic polynomial* of A. The equation det $(A - \lambda I) = 0$  is the *characteristic equation* of A.

Eigenvalues  $\lambda$  may be complex numbers, and the eigenvectors  $\boldsymbol{v}$  may have complex components, even for real matrices A.

To find the eigenvalues and eigenvectors, do the following:

- 1. Find the roots of the characteristic polynomial,  $det(A \lambda I) = 0$ . These are the eigenvalues.
- 2. For each eigenvalue  $\lambda$ , find  $N(A \lambda I)$ , known as the *eigenspace* associated to  $\lambda$ .

# 15.2.1 Example

Find the eigenvalues and eigenvectors of 
$$A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$
.



# 15.3 Simple properties

For a square matrix A:

**1.** A and  $A^T$  have the same eigenvalues.

**2.** A is singular if and only if  $\lambda = 0$  is an eigenvalue of A.

**3.** If  $\lambda$  is an eigenvalue of A, then  $\lambda^2$  is an eigenvalue of  $A^2$ , and  $1/\lambda$  is an eigenvalue of  $A^{-1}$  when A is non-singular.

4. If  $\lambda$  is an eigenvalue of A, then  $\lambda - m$  is an eigenvalue of A - mI, for any scalar m.

# 15.4 Eigenvectors corresponding to distinct eigenvalues are linearly independent

If  $\lambda_1, \lambda_2, \ldots, \lambda_k$  are distinct eigenvalues of A, with corresponding eigenvectors  $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ (such that  $\boldsymbol{v}_i$  corresponds to  $\lambda_i$ ), then the set of eigenvectors  $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$  is linearly independent. Notes.

## 16 Diagonalisation

By the end of this section, you should be able to answer the following questions:

- How do you find a matrix P which diagonalises a given matrix A?
- How do you determine if A is diagonalisable?
- What are two applications of diagonalisation?

A square matrix A is *diagonalisable* if there is a non-singular matrix P such that  $P^{-1}AP$  is a diagonal matrix. Here we consider the question: given a matrix, is it diagonalisable? If so, how do we find P?

The secret to constructing such a P is to let the columns of P be the eigenvectors of A. We immediately have that AP = PD, where D is a diagonal matrix with eigenvalues on the diagonal. We know from section 15.1 on page 111 that P is invertible if and only if the columns of P are linearly independent. Hence, we have the following result:

The  $n \times n$  matrix A is diagonalisable if and only if A has n linearly independent eigenvectors.

Is the matrix 
$$A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$
 diagonalisable?

#### 16.1 Similar matrices

Two matrices A and B are *similar* if there is a non-singular matrix P such that  $B = P^{-1}AP$ .

The statements "A is diagonalisable" and "A is similar to a diagonal matrix" are equivalent.

#### 16.1.1 Theorem (similar matrices)

Similar matrices have the same eigenvalues.

In fact, if  $B = P^{-1}AP$  and  $\boldsymbol{v}$  is an eigenvector of A corresponding to eigenvalue  $\lambda$ , then  $P^{-1}\boldsymbol{v}$  is an eigenvector of B corresponding to eigenvalue  $\lambda$ . This is because

$$B(P^{-1}\boldsymbol{v}) = (P^{-1}AP)P^{-1}\boldsymbol{v}$$
  
=  $P^{-1}(A\boldsymbol{v})$   
=  $P^{-1}(\lambda\boldsymbol{v})$   
=  $\lambda(P^{-1}\boldsymbol{v})$ 

#### 16.2 A closer look at the diagonal matrix

Let the matrix A be  $n \times n$  with n linearly independent eigenvectors  $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$  corresponding to eigenvalues  $\lambda_1, \ldots, \lambda_n$ . Let

$$P = (\boldsymbol{v}_1 | \dots | \boldsymbol{v}_n)$$

be the  $n \times n$  matrix whose columns are the eigenvectors. Then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

the diagonal matrix with the eigenvalues down the main diagonal. The important point here is the order in which the eigenvalues appear. They correspond to the order in which the associated eigenvectors appear in the columns of P.

#### 16.3 Diagonalisability

We know that an  $n \times n$  matrix A is diagonalisable if and only if A has n linearly independent eigenvectors.

Now say  $\lambda_1, \ldots, \lambda_m$  are *distinct* eigenvalues of A, with corresponding eigenvectors  $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ . Then we have also seen that  $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$  are linearly independent.

Hence if A is  $n \times n$  with n distinct eigenvalues, then A is diagonalisable.

The question remains, if A has fewer than n distinct eigenvalues, how do we know if A is diagonalisable?

16.3.1 Example

Let  $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ .

Easy to see the characteristic equation of both A and B is  $(2 - \lambda)(1 - \lambda)^2 = 0$ , so  $\lambda = 2, 1, 1$ .

### 16.4 Algebraic and geometric multiplicity

If we are only interested in finding out whether or not a matrix is diagonalisable, then we need to know the dimension of each eigenspace. There is one theorem (which we will not prove!) that states:

If  $\lambda_i$  is an eigenvalue, then the dimension of the corresponding eigenspace cannot be greater than the number of times  $(\lambda - \lambda_i)$  appears as a factor in the characteristic polynomial.

We often use the following terminology:

- The geometric multiplicity of the eigenvalue  $\lambda_i$  is the dimension of the eigenspace corresponding to  $\lambda_i$ .
- The algebraic multiplicity of the eigenvalue  $\lambda_i$  is the number of times  $(\lambda \lambda_i)$  appears as a factor in the characteristic polynomial.

The main result is the following:

A square matrix is diagonalisable if and only if the geometric and algebraic multiplicities are equal for every eigenvalue.

## 16.5 Applications of diagonalisability

#### 16.5.1 Systems of differential equations

For a system of coupled differential equations which can be written in matrix form as

$$\dot{\boldsymbol{x}} = A \boldsymbol{x}$$

(where  $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ ,  $\dot{\boldsymbol{x}} = (\dot{x}_1, \ldots, \dot{x}_n)^T$ ), if A can be diagonalised, say  $P^{-1}AP = D$  with D diagonal, then make the substitution  $\boldsymbol{x} = P\boldsymbol{y}$ . This yields

 $\dot{\boldsymbol{y}} = D\boldsymbol{y}$ 

which is easily solved.



### 16.5.2 Matrix powers

If A is diagonalisable, say  $P^{-1}AP = D$  with D diagonal, then

 $A^n = PD^nP^{-1}.$ 

This gives an easy way to calculate  $A^n$ .

#### 16.5.3 Linear systems

What can we say about the linear system (corresponding to a square matrix A)

 $A\boldsymbol{x} = \boldsymbol{b},$ 

if we know how to diagonalise A?

Notes.

# 17 Orthogonal matrices

The goal of this section is to investigate orthogonal matrices and to see their relationship with isometries, i.e. functions that preserve distance.

#### 17.1 Transpose of a transition matrix

Using the inner product on  $M_{2,2}(\mathbb{R})$  discussed in Section 8.5, we noted that

 $\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_{\beta} \cdot [\mathbf{v}]_{\beta},$ 

where  $\beta$  is the standard ordered basis for  $M_{2,2}(\mathbb{R})$ . This generalises to any finitedimensional real inner product space V with orthonormal basis  $\beta$ .

If  $\beta, \beta'$  are orthonormal bases for V, we then have

$$[\mathbf{u}]_{\beta} \cdot [\mathbf{v}]_{\beta} = \langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_{\beta'} \cdot [\mathbf{v}]_{\beta'}.$$

Since the transition matrix  $P_{\beta \to \beta'}$  relates the coordinate vectors as  $[\mathbf{u}]_{\beta'} = P_{\beta \to \beta'}[\mathbf{u}]_{\beta}$ , it follows that

$$[\mathbf{u}]_{\beta} \cdot [\mathbf{v}]_{\beta} = [\mathbf{u}]_{\beta'} \cdot [\mathbf{v}]_{\beta'} = (P_{\beta \to \beta'}[\mathbf{u}]_{\beta}) \cdot (P_{\beta \to \beta'}[\mathbf{v}]_{\beta}) = (P_{\beta \to \beta'}[\mathbf{u}]_{\beta})^T (P_{\beta \to \beta'}[\mathbf{v}]_{\beta})$$
$$= ([\mathbf{u}]_{\beta})^T (P_{\beta \to \beta'})^T P_{\beta \to \beta'}[\mathbf{v}]_{\beta} = [\mathbf{u}]_{\beta} \cdot ((P_{\beta \to \beta'})^T P_{\beta \to \beta'}[\mathbf{v}]_{\beta}).$$

Now, could it be that  $(P_{\beta \to \beta'})^T P_{\beta \to \beta'}$  is simply the identity matrix? It is! Since transition matrices are invertible, this means that

$$(P_{\beta \to \beta'})^T = (P_{\beta \to \beta'})^{-1}.$$

### 17.2 Orthogonal matrices

A square matrix Q with real entries is called **orthogonal** if it is invertible and

$$Q^{-1} = Q^T,$$

that is, if  $Q^T Q = Q Q^T = I$ . As shown in Section 17.1, transition matrices between orthonormal bases are examples of orthogonal matrices. Special cases are

$$Q_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \qquad \tilde{Q}_{\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

Orthogonal matrices have their name because of the following property. Let  $(\mathbf{v}_1 | \dots | \mathbf{v}_n)$  denote an  $n \times n$  matrix with column vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . Then,

 $(\mathbf{v}_1 | \dots | \mathbf{v}_n)$  is orthogonal  $\iff \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is an orthonormal set.

#### 17.3 Isometries

An **isometry** of  $\mathbb{R}^n$  is a function  $h : \mathbb{R}^n \to \mathbb{R}^n$  that preserves distances:

 $||h(\mathbf{u}) - h(\mathbf{v})|| = ||\mathbf{u} - \mathbf{v}||, \qquad \forall \, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$ 

As one can show, every isometry of  $\mathbb{R}^n$  can be uniquely written as a composition of a translation and an isometry that fixes **0** (the origin of  $\mathbb{R}^n$ ). Examples of isometries that fix **0** are rotations and reflections. Concretely, the matrix  $Q_{\theta}$  above generates rotations by  $\theta$  about the origin in the plane  $\mathbb{R}^2$ , while  $\hat{Q}_{\theta}$  generates reflections about the line through the origin, turned the angle  $\theta/2$ .

For a function  $h : \mathbb{R}^n \to \mathbb{R}^n$ , the following are equivalent:

- (i) h is an isometry and  $h(\mathbf{0}) = \mathbf{0}$ ;
- (ii)  $h(\mathbf{u}) \cdot h(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}, \quad \forall \, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$

## 17.4 Orthogonal matrices generate isometries

For  $Q \in M_{n,n}(\mathbb{R})$ , the following are equivalent:

- (i) Q is orthogonal;
- (ii)  $||Q\mathbf{v}|| = ||\mathbf{v}||, \quad \forall \mathbf{v} \in \mathbb{R}^n;$
- (iii)  $(Q\mathbf{u}) \cdot (Q\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}, \quad \forall \, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$

Thus, isometries of  $\mathbb{R}^n$  that fix **0** are generated by orthogonal matrices, and vice versa.

Notes.

#### Orthogonal diagonalisation 18

By the end of this section, you should be able to answer the following questions:

- What is a symmetric matrix?
- How do you diagonalise symmetric matrices?

Given an  $n \times n$  matrix A, we call A orthogonally diagonalisable if there exists an orthogonal matrix P such that  $P^{-1}AP = P^{T}AP$  is diagonal. To understand this, we first need to know what is meant by an orthogonal matrix.

#### 18.1**Orthogonal matrices**

Recall from the previous chapter that an *orthogonal* matrix is a real square matrix Qsuch that the columns of Q are mutually orthogonal unit vectors with respect to the Euclidean inner product (i.e.  $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$  if  $i \neq j$ , and  $||\boldsymbol{v}_i|| = 1$ ).

An orthogonal matrix is then a real square matrix Q such that  $Q^{-1} = Q^T$ . An immediate consequence of this is that  $det(Q) = \pm 1$ .

#### Symmetric matrices 18.2

A matrix A is symmetric if and only if  $A = A^{T}$ . Symmetric matrices are easy to

identify due to their "mirror symmetry" about the main diagonal. For example, we can tell by inspection that  $A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$  is symmetric.

**18.2.1** If A is real symmetric, then the eigenvectors corresponding to different eigenvalues are orthogonal.

Pro<u>of:</u>



#### 18.2.2 Real symmetric matrices are orthogonally diagonalisable

It is straightforward to show that if a matrix is orthogonally diagonalisable, then it is symmetric:

In fact, the converse is also true (although difficult to prove), giving us the amazing result:

An  $n \times n$  real matrix is orthogonally diagonalisable if and only if it symmetric.

The significance of this is that a symmetric matrix is *always* diagonalisable by an orthogonal matrix.

#### 18.2.3 Eigenvectors and eigenvalues

Here we state two results about any symmetric matrix A without proof:

- (1) All the eigenvalues of A are real;
- (2) A has n linearly independent eigenvectors.

#### 18.2.4 Example

Let  $A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$  (see previous examples). We already know the eigenvalues are -3, -1, -4 with corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1\\0\\-1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} 1\\2\\1 \end{pmatrix}, \ \boldsymbol{v}_3 = \begin{pmatrix} 1\\-1\\1 \end{pmatrix}.$$

Note that A is real symmetric, so  $v_1$ ,  $v_2$  and  $v_3$  should be pairwise orthogonal.

Notes.

## **19** Quadratic forms

By the end of this section, you should be able to answer the following questions:

- What is a quadratic form?
- How do you diagonalise quadratic forms?
- How can you use diagonalisation of two variable quadratic forms to identify conic sections?
- What are quadric surfaces?

This section presents a novel application of orthogonal diagonalisation as a way of identifying conic sections. We also mention the generalisation to three dimensions and how, in principle, we could identify quadric surfaces, although the details in this case can become quite messy.

The majority of this section is based on the section on quadratic forms in the MATH2001 recommended text "Elementary Linear Algebra (Applications Version)" by Anton and Rorres, pages 479–502.

#### 19.1 Definition

Consider *n* real variables  $x_1, x_2, \ldots, x_n$ . A function of the form  $\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$  is called

a quadratic form, where the  $a_{ij}$  are real constants.

For example, the most general quadratic form in the variables x and y is

$$Q(x,y) = ax^2 + by^2 + cxy.$$

In the three variables x, y and z, the most general quadratic form is

$$Q(x, y, z) = ax^{2} + by^{2} + cz^{2} + dxy + exz + fyz,$$

where in both cases a, b, c, d, e, f are all constants. It is possible to express quadratic forms in n variables as a matrix product  $\boldsymbol{v}^T A \boldsymbol{v}$ , where  $\boldsymbol{v}$  is a vector with the n variables as entries and A is a symmetric matrix.

The two variable quadratic form above can be expressed as

$$Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The three variable quadratic form given above can be written as

$$Q(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

As an exercise, trying verifying this by expanding out both expressions. Observe that in both cases the diagonal entries of the matrix are the coefficients of the square terms and the off-diagonal entries in the matrix are the coefficients of the cross-terms.

# **19.1.1** Give the matrix representation of the quadratic form $2x^2 + 6xy - 7y^2$ .



# 19.2 Diagonalising quadratic forms

Since we know we can always orthogonally diagonalise a symmetric matrix, if we do this to the symmetric matrix in the matrix representation of the quadratic form, we can reduce the quadratic form to a sum of square terms.

We shall demonstrate this by example:

19.2.1 Express  $-3x^2 - 2y^2 - 3z^2 + 2xy + 2yz$  exclusively as a sum of square terms.

#### **19.3** Quadratic equations and conic sections

We now restrict our attention to two dimensions, by investigating quadratic equations, which are equations of the form

$$ax^{2} + by^{2} + cxy + dx + ey + f = 0,$$

where  $a, b, c, d, e, f \in \mathbb{R}$ .

Graphs of quadratic equations are known as *conic sections*, because they can be realised as the intersection of a plane and a double cone in three dimensions. The most interesting of these are the so-called *non-degenerate* conic sections<sup>1</sup>. A non-degenerate conic section is in standard position relative to the coordinate axes if its equation can be expressed in one of the following forms:

• 
$$\frac{x^2}{k^2} + \frac{y^2}{l^2} = 1; k, l > 0,$$
  
•  $\frac{x^2}{k^2} - \frac{y^2}{l^2} = 1 \text{ or } \frac{y^2}{l^2} - \frac{x^2}{k^2} = 1; k, l > 0,$ 

• 
$$x^2 = ky$$
 or  $y^2 = kx; k \neq 0$ .

The key observation here is that conic sections in standard form have no cross-terms. Given a quadratic equation with cross-terms in the associated quadratic form, we can *change variables* to remove the cross-terms by orthogonal diagonalisation. Due to the defining property of rotation matrices, an orthogonal matrix P always corresponds to a rotation, provided det $(P) = 1 \pmod{-1}$ . Hence, we have the following.

Changing variables by orthogonal diagonalisation corresponds to a rotation of the coordinate axes. If P is the orthogonal (rotation) matrix, then the new coordinates (u, v) can be expressed in terms of the old coordinates (x, y) as

$$\left(\begin{array}{c} u\\ v\end{array}\right) = P^T \left(\begin{array}{c} x\\ y\end{array}\right).$$

Another important observation is that there is never an occurance of  $x^2$  and x in the standard form (or  $y^2$  and y). As a general rule, given a quadratic equation (even after changing variables from orthogonal diagonalisation), if we have terms such as  $x^2$ and x (or similar terms involving new variables) we can *complete the square* to be left with only a square term. We have the following.

Completing the square in a quadratic equation corresponds to translating (or shifting) the coordinate axes.

<sup>&</sup>lt;sup>1</sup>There are also *degenerate* (points, lines) and *imaginary* (without real graphs) conic sections.

In summary, to identify a quadratic equation as a conic section, we follow these steps:

1. Write the quadratic equation

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

in the matrix form  $\boldsymbol{x}^T A \boldsymbol{x} + K \boldsymbol{x} + f = 0$ , where  $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$  and  $K = \begin{pmatrix} d e \end{pmatrix}$ .

- 2. Find a matrix P that orthogonally diagonalises A, so  $A = PDP^{T}$ . You may need to swap columns of P to ensure that det(P) = 1 (and hence corresponds to a rotation).
- 3. Define new variables u, v such that  $\boldsymbol{v} = \begin{pmatrix} u \\ v \end{pmatrix} = P^T \boldsymbol{x} \Rightarrow \boldsymbol{x} = P \boldsymbol{v}.$
- 4. Substitute v into the matrix form of the equation, giving

$$\boldsymbol{v}^T D \boldsymbol{v} + K P \boldsymbol{v} + f = 0.$$

- 5. Complete the square if required. This is necessary if  $u^2$  and u are both present (or  $v^2$  and v). This defines a new set of variables s, t by translating u, v. The translations will be of the form  $s = \alpha u + \beta$ ,  $t = \gamma v + \delta$ .
- 6. If it is a non-degenerate conic, the final equation in s and t should be a conic section in standard form.

## 19.3.1 Describe the conic whose equation is $x^2 + y^2 + 2xy - 3x - 5y + 4 = 0$ .

#### **19.4** Further reading: quadric surfaces

It turns out we can similarly use orthogonal diagonalisation of  $3 \times 3$  matrices to simplify and ultimately identify surfaces whose general equation is of the form

$$ax^{2} + by^{2} + cz^{2} + 2dxy + 2exz + 2fyz + gx + hy + iz + j = 0,$$

where a, b, c, d, e, f are never all zero. Note that we can rewrite the equation in matrix form as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + j = 0.$$

If we then orthogonally diagonalise the  $3 \times 3$  matrix, then complete any squares that are left over, we end up being able to identify the surface as one of the following forms:

•  $\frac{x^2}{l^2} + \frac{y^2}{m^2} + \frac{z^2}{n^2} = 1$ , •  $z^2 = \frac{x^2}{l^2} + \frac{y^2}{m^2}$ , •  $\frac{x^2}{l^2} + \frac{y^2}{m^2} - \frac{z^2}{n^2} = 1$ , •  $z = \frac{x^2}{l^2} + \frac{y^2}{m^2}$ , •  $\frac{z^2}{l^2} - \frac{x^2}{m^2} - \frac{y^2}{n^2} = 1$ , •  $z = \frac{y^2}{m^2} - \frac{x^2}{l^2}$ .

As in the two dimensional case, the orthogonal diagonalisation has the effect of rotating the axes, provided the orthogonal matrix P has det P = 1, which we can choose by carefully ordering the columns. Completing the square has the effect of shifting the axes.

You should be aware that these techniques are available in order to simplify and identify algebraic expressions representing surfaces. The three dimensional case can often become quite complicated. You will not be expected to identify quadric surfaces in an exam. Notes.

# 20 Complex matrices

By the end of this section, you should be able to answer the following questions:

- What are unitary, Hermitian and normal matrices?
- Given a complex matrix, determine if it can be unitarily diagonalised, and if so, diagonalise it.

Unitary and Hermitian matrices are complex analogues of orthogonal  $(A^{-1} = A^T)$ and symmetric  $(A = A^T)$  real matrices respectively.

In order to define these matrices, we need the following.

### 20.1 Definition (conjugate transpose)

Let A be a complex matrix. The conjugate transpose of A, denoted  $A^*$ , is given by  $(\overline{A})^T$ , where  $\overline{A}$  is the matrix whose entries are complex conjugates of the corresponding entries of A.

Note that if A is real,  $A^* = A^T$ .

#### 20.1.1 Example

Let  $A = \begin{pmatrix} 3+7i & 0\\ 2i & 4-i \end{pmatrix}$ . Write down the conjugate transpose of A.
#### 20.2 Unitary matrices

A complex matrix A is said to be *unitary* if  $A^{-1} = A^*$ . Compare this definition with that of real orthogonal matrices.

Recall that a real matrix is orthogonal if and only if its columns form an orthonormal set of vectors. For complex matrices, this property characterises unitary matrices. In this case however, we must use the complex inner product.

#### 20.3 Complex inner product

Recall that in  $\mathbb{R}^n$  the inner (or dot) product of two vectors

$$oldsymbol{u} = \left(egin{array}{c} u_1 \ u_2 \ dots \ u_n \end{array}
ight), \quad oldsymbol{v} = \left(egin{array}{c} v_1 \ v_2 \ dots \ v_n \end{array}
ight)$$

is given by

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_1v_1+u_2v_2+\cdots+u_nv_n$$

and the length (a real number!) of  $\boldsymbol{u}$  by

$$||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}.$$

These definitions are unsuitable for vectors in  $\mathbb{C}^n$ .

To demonstrate, consider the vector  $\boldsymbol{u} = (i, 1)$  in  $\mathbb{C}^2$ . Using the above expression for length, we would obtain  $||\boldsymbol{u}|| = \sqrt{i^2 + 1} = 0$ , so  $\boldsymbol{u}$  would be a non-zero vector with length 0.

Instead, we introduce the complex inner product

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_1\overline{v}_1+u_2\overline{v}_2+\cdots+u_n\overline{v}_n,$$

where as usual  $\overline{v}_i$  denotes the complex conjugate of  $v_i$ . In matrix notation, we can write this as  $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v}^* \boldsymbol{u}$ . Note the length of a complex vector is always a real number.

So now we understand what is meant by the following statement: Columns of a unitary matrix form an orthonormal set with respect to the complex inner product.

#### 20.4 Hermitian (self-adjoint) matrices

A complex matrix A is called *Hermitian* (or *self-adjoint*) if  $A = A^*$ .

As with symmetric matrices, we can recognise a Hermitian matrix by inspection. See if you can see the pattern in the following  $2 \times 2$ ,  $3 \times 3$  and  $4 \times 4$  Hermitian matrices.

$$\begin{pmatrix} a_{11} & a_{12} + ib_{12} \\ a_{12} - ib_{12} & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} + ib_{12} & a_{13} + ib_{13} \\ a_{12} - ib_{12} & a_{22} & a_{23} + ib_{23} \\ a_{13} - ib_{13} & a_{23} - ib_{23} & a_{33} \end{pmatrix},$$
$$\begin{pmatrix} a_{11} & a_{12} + ib_{12} & a_{13} + ib_{13} & a_{14} + ib_{14} \\ a_{12} - ib_{12} & a_{22} & a_{23} + ib_{23} & a_{24} + ib_{24} \\ a_{13} - ib_{13} & a_{23} - ib_{23} & a_{33} & a_{34} + ib_{34} \\ a_{14} - ib_{14} & a_{24} - ib_{24} & a_{34} - ib_{34} & a_{44} \end{pmatrix},$$

where  $a_{ij}, b_{ij} \in \mathbb{R}$ . Note in particular that the diagonal entries are real numbers.

One of the most significant results on Hermitian matrices is that their eigenvalues are real.

#### 20.4.1 Proof that Hermitian matrices have real eigenvalues

Let  $\boldsymbol{v} \in \mathbb{C}^n$  be an eigenvector of the Hermitian matrix A, with corresponding eigenvalue  $\lambda$ . In other words,

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.\tag{12}$$

In what follows, we use the fact that  $(AB)^* = B^*A^*$  which holds since the same is true for matrix transposition.

We multiply (12) from the left by  $\boldsymbol{v}^*$  (treat  $\boldsymbol{v}$  as an  $n \times 1$  complex matrix) to obtain

$$\boldsymbol{v}^* A \boldsymbol{v} = \boldsymbol{v}^* (\lambda \boldsymbol{v}) = \lambda (\boldsymbol{v}^* \boldsymbol{v}). \tag{13}$$

Also note that

$$(\boldsymbol{v}^*A\boldsymbol{v})^* = \boldsymbol{v}^*A^*(\boldsymbol{v}^*)^* = \boldsymbol{v}^*A\boldsymbol{v}.$$

In other words,  $v^*Av$  is also Hermitian. Since it evaluates to be a  $1 \times 1$  matrix, and all Hermitian matrices have real numbers on their diagonal, this means that  $v^*Av$  is a real number.

The quantity  $v^*v$  is precisely the complex inner product of v with itself as we have already seen, which is also a real number.

Therefore equation (13) is of the form

$$x = \lambda y, \quad x, y \in \mathbb{R},$$

from which we must conclude that  $\lambda$  is real.

One consequence of this result is that a real symmetric matrix has real eigenvalues, since every real symetric matrix is Hermitian. This result was stated on page 133 but not proved.

## 20.5 Unitary diagonalisation

We have seen that real symmetric matrices are orthogonally diagonalisable. There is an analogous concept for complex matrices.

A square matrix A with complex entries is said to be unitarily diagonalisable if there is a unitary matrix P such that  $P^*AP$  is diagonal.

It is natural to consider which matrices are unitarily diagonalisable. The answer lies in a more general class of matrix.

#### 20.6 Normal matrices

A square complex matrix is called normal if it commutes with its own conjugate transpose, ie, if  $AA^* = A^*A$ .

Normal matrices are generally more difficult to identify by inspection. However, we have some classes of matrices which are normal:

- unitary,
- Hermitian,
- real skew-symmetric (satisfying  $A^T = -A$ ),
- any diagonal matrix,
- others?

We make a note that real normal  $2 \times 2$  matrices are either symmetric or of the form  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$  (which include the skew-symmetric examples).

A class of matrix which is not generally normal is the class of complex symmetric matrices.

## 20.6.1 Example

Classify the matrix  $A = \begin{pmatrix} 1 & 1+i \\ 1+i & -i \end{pmatrix}$ .

## 20.7 Normal = unitarily diagonalisable

The main result we have is completely analogous to the real case of orthogonal diagonalisation and symmetric matrices on page 133. We will not prove this result.

An  $n \times n$  complex matrix is unitarily diagonalisable if and only if it normal.

#### 20.7.1 Example

If possible, diagonalise the matrix  $\begin{pmatrix} 6 & 2+2i \\ 2-2i & 4 \end{pmatrix}$ .

Notes.

## 21 Multivariable Taylor series

By the end of this section you should be able to write down Taylor series of smooth, scalar-valued functions of n variables.

Let the function  $f : \mathbb{R} \to \mathbb{R}$  be infinitely differentiable at  $a \in \mathbb{R}$ . Then, the Taylor series of f(x) is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

A common reason for calculating the Taylor series of a function about a point x = a is that its partial sums provide (polynomial) approximations to the function near the point a. Here, we want to understand how this extends to functions of more than one variable.

Before we continue, it is recalled that Taylor series need not be convergent, and even if the Taylor series of f(x) does converge, its limit need not equal f(x) (in which case f is known as *non-analytic*). To accommodate these situations, we may write

$$f(x) \approx f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots$$

#### 21.1 Hessian matrix

For  $n \in \mathbb{N}$ , let  $D \subseteq \mathbb{R}^n$  and  $f : D \to \mathbb{R}$  such that all second-order partial derivatives of f,

$$f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \qquad i, j = 1, \dots, n,$$

are continuous on D. The corresponding **Hessian** is then defined as the  $n \times n$  matrix

$$H = \begin{pmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & & \vdots \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix}.$$

According to Clairaut's theorem,  $f_{x_ix_j} = f_{x_jx_i}$  for all i, j = 1, ..., n, so  $H^T = H$ , meaning that H is a symmetric matrix. To specify the dependence on f and  $\mathbf{x} = (x_1, ..., x_n)$ , one occasionally writes  $H(f)(\mathbf{x})$  or  $H_f(\mathbf{x})$ .

## **21.2** Example: *H* of $f(x, y) = x^3y + 2y$ at (1,2)

## **21.3** Taylor series of f(x, y)

Let  $D \subseteq \mathbb{R}^2$  and  $f: D \to \mathbb{R}$  such that f and all its (higher-order) partial derivatives are continuous on D. Such a function is often called **smooth**. Here, we will examine the behaviour of f near (x, y) = (a, b) by reducing the problem to a related one-variable problem we know how to handle.

Let  $\mathbf{v} = (h, k) \neq (0, 0)$  be small enough such that  $(a + ht, b + kt) \in D$  for  $t \in [-1, 1]$ . The function

$$F(t) = f(a + ht, b + kt), \qquad t \in [-1, 1],$$

then describes how f(x, y) behaves along the line parameterised as

$$x(t) = a + ht, \qquad y(t) = b + kt.$$

Understanding this behaviour near t = 0 will give us insight into how f behaves near (x, y) = (a, b). Geometrically, we can think of it as follows:

Algebraically, we have the following:

Up to higher-order terms in t, we thus have

$$f(a+ht,b+kt) \approx f(a,b) + [f_x(a,b)h + f_y(a,b)k)]t + [f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2]\frac{t^2}{2} + \dots$$

Since (x, y) = (a + ht, b + kt) implies that ht = x - a and kt = y - b, we have  $h^2t^2 = (x - a)^2$ ,  $hkt^2 = (x - a)(y - b)$  and  $k^2t^2 = (y - b)^2$ , so

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2} [f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2] + \dots$$

Terms corresponding to higher powers of t involve factors of the form  $(x-a)^p(y-b)^q$ , where  $p+q>2, p,q \in \mathbb{N}$ .

Recalling the notion of *quadratic forms* from the lecture workbook, we can rewrite the first few terms of the Taylor series as follows:

Setting

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix},$$

we have

$$\begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix} = \nabla f(\mathbf{x}_0), \qquad \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix} = H_f(\mathbf{x}_0).$$

This allows us to write the first few terms of the Taylor series in the compact form

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0))^T (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2} (\mathbf{x} - \mathbf{x}_0)^T H_f(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0) + \dots$$

Its vectorial expression indicates quite clearly how it extends to higher dimensions.

**21.4** Example:  $f(x,y) = \frac{\sin(x-y^2)}{\sqrt{y+1}}$  about (0,0)

## 21.5 Taylor series in any dimension

Let f be a smooth scalar-valued function on  $D \subseteq \mathbb{R}^n$ , and let  $\mathbf{h} \in \mathbb{R}^n$  such that  $\mathbf{x} + \mathbf{h}t \in D$  for  $t \in [-1, 1]$ . Then, the Taylor series of  $f(\mathbf{x} + \mathbf{h})$  about  $\mathbf{x}$  is

$$f(\mathbf{x} + \mathbf{h}) \approx \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (\mathbf{h} \cdot \nabla)^{\ell} f(\mathbf{x}) = f(\mathbf{x}) + \mathbf{h} \cdot \nabla f(\mathbf{x}) + \frac{1}{2} (\mathbf{h} \cdot \nabla)^{2} f(\mathbf{x}) + \dots$$

## 22 Critical points in *n*-dimensions

This chapter brings together a great deal of what we have studied so far in this course. The goal is to be able to classify critical points of functions of any number of variables.

Recall Taylor series of a smooth function f in n variables about a point  $\mathbf{x} = \mathbf{x}_0$  is given by;

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0))^T (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2} (\mathbf{x} - \mathbf{x}_0)^T H(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0) + \langle \text{ higher order terms} \rangle,$$

where 
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
,  $H(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}_0) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}_0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}_0) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}_0) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}_0) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{x}_0) \end{pmatrix} = H(\mathbf{x}_0)^T$ 

i.e.  $H(\mathbf{x}_0)$  is a real symmetric matrix.

#### 22.1 Classification of critical points in *n* dimensions

In the following, let  $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ .

**Definition 1.** A point  $\mathbf{x}_0$  is said to be a *critical point* if  $\nabla f(\mathbf{x}_0) = \mathbf{0}$  or  $\nabla f(\mathbf{x}_0)$  is undefined.

**Definition 2.** A critical point  $\mathbf{x}_0$  satisfying  $\nabla f(\mathbf{x}_0) = \mathbf{0}$  is a *local maximum (local minimum)* if there exists some  $\epsilon > 0$  such that  $f(\mathbf{x}_0) \ge f(\mathbf{x})$   $(f(\mathbf{x}_0) \le f(\mathbf{x}))$  for all  $\mathbf{x}$  such that  $||\mathbf{x} - \mathbf{x}_0|| < \epsilon$ .

**Definition 3.** A critical point  $\mathbf{x}_0$  satisfying  $\nabla f(\mathbf{x}_0) = \mathbf{0}$  is a *saddle point* if it is neither a local maximum nor a local minimum.

## 22.3 Critical points by Taylor series

In MATH1052: We used the "Second derivative test" for functions of two variables.

In MATH2001/7000: We consider a variant of this test that generalises easily to higher dimensions.

Let  $\mathbf{x}_0$  be a critical point satisfying  $\nabla f(\mathbf{x}_0) = \mathbf{0}$  $\implies$  Taylor series about  $\mathbf{x}_0$  is

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \frac{1}{2} (\mathbf{x} - \mathbf{x}_0)^T H(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0) + \langle \text{ higher order terms } \rangle.$$

Without loss of generality, we take  $\mathbf{x}_0 = \mathbf{0}$  (i.e. by shifting/translating variables if necessary).

We have,

$$f(\mathbf{x}) = f(\mathbf{0}) + \frac{1}{2}\mathbf{x}^T H\mathbf{x} + \langle \text{ higher order terms } \rangle.$$

Here  $H = H(\mathbf{0})$ . Thus, the behaviour about  $\mathbf{0}$  (i.e the critical point) depends on this second order term.

Observe that H is real symmetric  $\implies$  H is orthogonally diagonalisable, i.e. there exists an orthogonal matrix P such that  $P^T H P = D$  with some diagonal matrix D.

It follows that

$$\mathbf{x}^T H \mathbf{x} = (\mathbf{x}^T P) D(P^T \mathbf{x}) = \mathbf{y}^T D \mathbf{y}$$

(i.e. diagonalisation suggests set  $P^T \mathbf{x} = \mathbf{y}$ ).

The critical point is still at  $\mathbf{y} = \mathbf{0}$ , because  $P^T \mathbf{0} = \mathbf{0}$ .

Let F denote the function f expressed in this new coordinate system i.e.  $F(\mathbf{y}) = f(\mathbf{x}(\mathbf{y}))$ .

$$\implies F(\mathbf{y}) = f(\mathbf{0}) + \frac{1}{2}\mathbf{y}^T D\mathbf{y} + \langle \text{ higher order terms } \rangle$$
$$= f(\mathbf{0}) + \frac{1}{2} \left( \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \right)$$
$$+ \langle \text{ higher order terms } \rangle,$$

where  $\mathbf{y} = \left( egin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} 
ight).$ 

There are four cases to consider:

**22.4** Example:  $Q = ax^2 + bxy + cy^2$ 

Notes.

## 23 Introduction to double integrals, volume below a surface

By the end of this section, you should be able to answer the following questions:

- What is the definition of volume below a surface?
- What is the definition of a double integral?
- How are the two related?
- What is an iterated integral?

Recall that if y = f(x), the area under the curve over the interval I = [a, b] is

$$\int_{I} f(x) dx = \lim \sum_{i=1}^{n} f(x_{i}^{*})(x_{i} - x_{i-1})$$

where  $x_i^* \in [x_i, x_{i-1}]$ .

## 23.1 Double integrals

Suppose we have a surface z = f(x, y) above a planar region R in the x-y plane.



Figure 10: What is the volume V under the surface?

Before moving onto general regions, we start by considering the case where R is a rectangle. That is,

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, \ c \le y \le d\}.$$



Start by dividing R into subrectangles by dividing the interval [a, b] into m subintervals  $[x_{i-1}, x_i]$ , each of width  $\Delta x = \frac{b-a}{m}$  and [c, d] into n subintervals  $[y_{j-1}, y_j]$  of equal width  $\Delta y = \frac{d-c}{n}$ .

Combining these gives a rectangular grid  $R_{ij}$  with subrectangles each of area  $\Delta A = \Delta x \Delta y$ .

In each subrectangle take any point  $P_{ij}$  with co-ordinates  $(x_{ij}^*, y_{ij}^*)$ .

The volume of the box with base the rectangle  $\Delta A$  and height the value of the function f(x, y) at the point  $P_{ij}$  (so the box touches the surface at a point directly above  $P_{ij}$  - see figure 11) is

$$V_{ij} = f(x_{ij}^*, y_{ij}^*) \Delta A.$$

Then for all the subrectangles we have an approximation to the required volume V:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A,$$

the double Riemann sum.



Figure 11: The rectangular box whose volume is  $z^*\Delta A$ .

Let  $\Delta x \to 0$  and  $\Delta y \to 0$ , ie  $m \to \infty$  and  $n \to \infty$ , then we *define* the volume to be

$$V = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A,$$

if the limits exist and we write this as

$$\iint_R f(x,y) dA.$$

We call f integrable if the limits exist. Note that every continuous function is integrable.

23.2 Properties of the double integral

(i) 
$$\iint_{R} (f \pm g) dA = \iint_{R} f dA \pm \iint_{R} g dA$$

(ii) 
$$\iint_R cf dA = c \iint_R f dA$$

(iii) 
$$\iint_{R} f dA = \iint_{R_1} f dA + \iint_{R_2} f dA$$

(iv) If  $f(x,y) \ge g(x,y)$  for all  $(x,y) \in R$  then

$$\iint_R f dA \ge \iint_R g dA$$

## 23.3 Iterated integrals

We define  $\int_{c}^{d} f(x, y) dy$  to mean that x is fixed and f(x, y) is integrated with respect to y from y = c to y = d. So

$$A(x) = \int_c^d f(x,y) dy$$

is a function of x only.

If we now integrate A(x) with respect to x from x = a to x = b we have

$$\int_{a}^{b} A(x)dx = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y)dy \right] dx$$
$$= \int_{a}^{b} \int_{c}^{d} f(x,y) dy dx$$

This is called an iterated integral.

# **23.3.1 Example: evaluate** $\int_0^2 \int_1^3 x^2 y \, dy \, dx$



Now try integrating the other way around:







Figure 12: We have just calculated the volume of the solid outlined above.

Notes.

## 24 Fubini's theorem, volume by slabs

By the end of this section, you should be able to answer the following questions:

- What is Fubini's theorem?
- How is the double integral related to the iterated integral?
- How do you estimate the volume below a surface using slabs?

#### 24.1 Fubini's theorem

If f(x, y) is integrable on the rectangle

$$R = \left\{ (x, y) | a \le x \le b, \ c \le y \le d \right\},\$$

then

$$\iint_{R} f(x,y) dA = \int_{a}^{b} \int_{c}^{d} f(x,y) dy dx$$
$$= \int_{c}^{d} \int_{a}^{b} f(x,y) dx dy$$

**24.2 Example: evaluate**  $\iint_{R} (x^{2} + y^{2}) dA$  where  $R = \{(x, y) | 0 \le x \le 2, 0 \le y \le 1\}$ 





Figure 13: A representation of the volume in example 24.2.

#### 24.3 Interpreting Fubini's theorem in terms of volume

Fubini's theorem is the key result that tells us how to evaluate a double integral. We can see the relation between the iterated integral and the double integral by considering an alternative way of calculating the volume below a surface.

Suppose we want to find the volume below the surface  $z = x^2 y$  above the square region  $0 \le x \le 8$  and  $0 \le y \le 4$ .

A natural way to solve this problem is to break the region up into slabs of equal depth  $\Delta y = y_{j+1} - y_j$  located at  $y_j$ , and add up the volume of the slabs

$$V \approx \sum_{j} \Delta V,$$

where  $\Delta V$  the volume of the *j*th slab. Figure 14 below shows two ways of doing this using four slabs in each case. The left diagram follows the method outlined here, taking slabs of thickness  $\Delta y$ .



Figure 14: Two ways of approximating the volume under  $z = x^2 y$  using four slabs.

If the slab is very thin (i.e.  $\Delta y \ll 1$ ) then the volume of each slab is

$$\Delta V \approx$$
 Area of slab  $\times$  Depth  $= C(y_i) \Delta y$ .

Here  $C(y_j)$  is the area of the slab at the location  $y_j$  (and the result will depend on  $y_j$ !). From one-dimensional calculus we know exactly that

$$C(y_j) = \int_0^8 f(x, y_j) dx \quad y_j \text{ constant.}$$

It is easy to compute this as a regular integral since  $y_j$  does not vary with x. Putting all this together

$$V \approx \sum_{j} \Delta V_{j} \approx \sum_{j} C(y_{j}) \Delta y.$$

As the slabs become thinner and thinner  $(\Delta y \to 0)$  the approximation becomes more accurate and we can replace the summation by an integral<sup>2</sup>

$$V = \int_{0}^{4} C(y) dy = \int_{0}^{4} \left( \int_{0}^{8} f(x, y) dx \right) dy$$

Note that the y is held constant in the inner integral.

A similar argument can be applied by considering slabs of depth  $\Delta x$ , located at  $x_j$ . In other words, take slabs that are parallel to the y-z plane.

24.4 Example: find the volume of the solid bounded by the elliptic paraboloid  $x^2 + 2y^2 + z = 16$ , the planes x = 2 and y = 2, and the three coordinate planes.

 $<sup>^2\</sup>mathrm{Recall}$  that is in fact the definition of an integral





Figure 15: The volume of the solid of example 24.4 is below the surface  $z = 16 - x^2 - 2y^2$ and above the x-y plane as shown.

# **24.5** Special case when f(x,y) = g(x)h(y).

In this case we can separate the integral as follows.

$$\iint_{R} f(x,y) dA = \int_{c}^{d} \int_{a}^{b} g(x)h(y) dx dy$$
$$= \int_{a}^{b} g(x) dx \int_{c}^{d} h(y) dy$$

**24.5.1** Example:  $\iint_R \sin x \cos y \ dA$  where  $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ 





Figure 16: The volume calculated in example 24.5.1 is outlined above.

Notes.

## 25 Integrals over general regions

By the end of this section, you should be able to answer the following questions:

- How can you identify type I and II regions?
- How do you evaluate a double integral over type I and II regions?
- How can you evaluate a double integral over a more general region comprising finitely many type I and II regions?
- What is meant by net volume below a surface?

To find the double integral over a general region D instead of just a rectangle we consider a rectangle which encloses D and define

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D\\ 0, & \text{if } (x,y) \in R \text{ but } \notin D \end{cases}$$

then

$$\iint_{D} f(x,y)dA = \iint_{R} F(x,y)dA$$

and we can proceed as before. It is possible to show that F is integrable if the boundary of D is bounded by a finite number of smooth curves of finite length. Note that F may still be discontinuous at the boundary of D.

## 25.1 Type I regions

A plane region D is of type I if it lies between the graph of two continuous functions of x. That is  $D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}.$ 



Figure 17: Type I regions are generally bounded by two constant values of x and two functions of x.

In practice, to evaluate  $\iint_D f(x, y) dA$  where D is a region of type I we have

$$\iint_{D} f(x,y) dA = \int_{a}^{b} \int_{g_{1}(x)}^{g_{2}(x)} f(x,y) \, dy \, dx.$$



Figure 18: Some more examples of type I regions.

**25.1.1** Example: find  $\iint_D (4x + 10y) dA$  where *D* is the region between the parabola  $y = x^2$  and the line y = x + 2.



Figure 19: The volume of example 25.1.1 is outlined above.





Figure 20: Type II regions are generally bounded by two constant values of y and two functions of y.

A plane region is of type II if it can be expressed by

$$D = \{(x, y) | c \le y \le d, h_1(y) \le x \le h_2(y) \}.$$

In practice, to evaluate  $\iint_D f(x, y) dA$  where D is a region of type II we have

$$\iint_{D} f(x,y) dA = \int_{c}^{d} \int_{h_{1}(y)}^{h_{2}(y)} f(x,y) \, dx \, dy.$$



Figure 21: Some more examples of type II regions.

**25.2.1** Example: evaluate  $\iint_D xy \ dA$  where *D* is the region bounded by the line y = x - 1 and the parabola  $y^2 = 2x + 6$ .



Figure 22: The volume of example 25.2.1 is outlined above. Note carefully that the surface is *above* the x-y plane only in the quadrants where x, y > 0 and x, y < 0. For x and y values in the other two quadrants, the surface is *below* the x-y plane. Hence in this example we are calculating the "net volume" lying above the x-y plane.


25.3 Express D as a union of regions of type I or type II and expand the integral  $\iint_{D} f(x,y) dA$ , for some integrable function f. Notes.

# 26 Interchanging order of integration

By the end of this section, you should be able to answer the following questions:

- How do you change the order of integration in a double integral?
- when might it be necessary to change the order of integration in a double integral?

It is often possible to represent a type I region as a union of type II regions, or a type II region as a union of type I regions. Why would we want to do that? In some cases, it may only be possible to integrate a function one way but not the other. In this section, we investigate this idea more closely.

26.1 Find the volume under the paraboloid  $z = x^2 + y^2$  above the region *D*, where *D* is bounded by  $y = x^2$  and y = 2x. Do the problem twice, first by taking *D* to be a type I region, then by taking *D* to be type II.





Figure 23: This volume can be calculated by treating the region in the x-y plane as either type I or II as seen in example 26.1.

In the following example, we see how it is sometimes necessary to change the order of integration in order to evaluate the integral.

**26.2 Example: Find** 
$$\int_0^1 \int_x^1 \sin(y^2) \, dy \, dx$$



Figure 24: The volume described in example 26.2.



Notes.

## 27 Review of applications: volume, area

Main points:

- This section is a review of applications of the double integral such as calculating net volume and area in the plane.
- By this stage you should be comfortable with using a double integral to calculate the net volume below a surface.
- You should know how to find the area of a general region in the plane.

When the regions are more difficult, it is a good idea to draw two diagrams - the 3-D diagram with the x-y-z axes and the 2-D one of the region in the x-y plane.

27.1 Example: Find the volume of the tetrahedron bounded by the planes x + 2y + z = 2, x = 2y, x = 0 and z = 0.



Figure 25: You should be able to reproduce a diagram like this one as an aid to determining the bounds of integration.



#### 27.2 Area

Note that if we take f(x, y) = 1, we have  $\iint_D 1 \ dA = \text{area of the region } D$ .

27.3 Find the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 





Notes.

## 28 Double integrals in polar coordinates

By the end of this section, you should be able to answer the following questions:

- What is the relationship between polar coordinates and rectangular coordinates?
- How do you transform a double integral in rectangular coordinates into one in terms of polar coordinates?
- What is the Jacobian of the transformation?

For annular regions with circular symmetry, rectangular coordinates are difficult. It can be more convenient to use *polar coordinates*.

The following diagram explains the relationship between the polar variables  $r, \theta$  and the usual rectangular ones x, y.



For polar coordinates, we have

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Consider the volume of a solid beneath a surface z = f(x, y) and above a circular region in the x-y plane.

We divide the region into a polar grid as in the following diagram:



We first approximate the area of each polar rectangle as a regular rectangle. We do this as follows. Choose a point P inside each polar rectangle in the polar grid. Let  $P = (x^*, y^*)$  or in polar coordinates  $P = (r^*, \theta^*)$ , where

$$x^* = r^* \cos \theta^*, \quad y^* = r^* \sin \theta^*.$$

The area of the polar rectangle containing P can be approximated as  $r^*\Delta\theta\Delta r$ . Therefore the volume under the surface and above each polar rectangle can be approximated as

vol. one box 
$$\approx r^* \Delta \theta \Delta r f(r^* \cos \theta^*, r^* \sin \theta^*)$$
.

Here  $f(r^* \cos \theta^*, r^* \sin \theta^*)$  is the value of the function at the point P, which is also the height of the box used in the approximation. To obtain an approximation for the entire

volume below the surface, we sum over the entire polar grid:

vol. 
$$\approx \sum_{\text{(polar grid)}} r^* \Delta \theta \Delta r f(r^* \cos \theta^*, r^* \sin \theta^*)$$
  
 $\Rightarrow \text{ vol.} = \lim_{\Delta r, \Delta \theta \to 0} \sum_{\text{(polar grid)}} r^* \Delta \theta \Delta r f(r^* \cos \theta^*, r^* \sin \theta^*)$   
 $= \iint_D f(r \cos \theta, r \sin \theta) r \ d\theta \ dr.$ 

The double integral in rectangular coordinates is then transformed as follows:  $\iint_R f(x,y) \ dx \ dy = \iint_S f(r\cos\theta, r\sin\theta)r \ dr \ d\theta.$ 

28.1 Example: Find  $\iint_D e^{-(x^2+y^2)} dx dy$  where D is the region bounded by the circle  $x^2 + y^2 = R^2$ .





28.2 Example: Find the volume of the solid bounded by the plane z = 0 and the paraboloid  $z = 1 - x^2 - y^2$ .





28.3 Find the volume of the solid that lies under the paraboloid  $z = x^2 + y^2$  and inside the cylinder  $x^2 + y^2 = 2x$ , for  $z \ge 0$ .





Notes.

### 29 Mass, centre of mass and moments

By the end of this section, you should be able to answer the following questions:

- How can we use a double integral to find the mass of a two dimensional object if the density function is known?
- How do we use double integrals to locate the centre of mass of such an object?
- How do we calculate the moments of such an object about the coordinate axes?

Ultimately we want to find a point P on which a thin plate of any given shape balances horizontally. Such a point is called the centre of mass of the plate.



Consider a rod of negligible mass balanced on a fulcrum. The rod has masses  $m_1$ and  $m_2$  at either end, which are a distance  $d_1$  and  $d_2$  respectively from the fulcrum. Because the rod is balanced, we have (thanks to Archimedes) the relationship

$$m_1d_1 = m_2d_2.$$

Now suppose the rod lies on the x-axis with  $m_1$  at  $x = x_1$ ,  $m_2$  at  $x = x_2$  and the centre of mass at  $\overline{x}$ .



In this case we can write  $d_1 = \overline{x} - x_1$  and  $d_2 = x_2 - \overline{x}$ , so Archimedes' relationship can be expressed

$$m_1(\overline{x} - x_1) = m_2(x_2 - \overline{x}) \Rightarrow \overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

The numbers  $m_1x_1$  and  $m_2x_2$  are called the *moments* of the masses  $m_1$  and  $m_2$  respectively.



In general, a one dimensional system of n "particles" with masses  $m_1, \ldots, m_n$  located at  $x = x_1, \ldots, x_n$  has its centre of mass located at

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M}{m}$$

where  $m = \sum m_i$  is the total mass of the system and the sum of the individual moments  $M = \sum m_i x_i$  is called the moment of the system (with respect to the origin).

Now suppose the rod (which has length l) has mass which is distributed according to the (integrable) density function (mass/unit length)

$$\rho(x) = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x}.$$

Consider a small strip of width  $\Delta x$  containing the point  $x^*$ . The mass of this strip can be approximated by  $\rho(x^*)\Delta x$ . Now cut the rod into *n* strips, and in the same way as above determine (approximately) the mass of each strip. To obtain an approximation for the total mass *m* of the rod, just add the masses of each *n* strips:

$$m \approx \sum_{i=1}^{n} \rho(x_i^*) \Delta x_i.$$

To obtain a precise expression for the mass, we take the limit of this sum as  $n \to \infty$ . In other words,

$$m = \int_0^l \rho(x) dx.$$

We have a similar construction for the moment of the system. Consider the moment of each strip  $\approx x_i^* \rho(x_i^*) \Delta x_i$ . If we add these, we obtain an approximate expression for the moment of the system:

$$M \approx \sum_{i=1}^{n} x_i^* \rho(x_i^*) \Delta x_i$$

Taking the limit as  $n \to \infty$  we obtain an expression for the moment of the system about the origin:

$$M = \int_0^l x \rho(x) dx.$$

The centre of mass is located at  $\overline{x} = M/m$ . Now let's generalize this to two dimensions. Suppose the lamina occupies a region D in the x-y plane and its density (in units of mass/unit area) is given by an integrable function  $\rho(x, y)$ . In other words,

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A},$$

where  $\Delta m$  and  $\Delta A$  are the mass and area of a small rectangle containing the point (x, y), and the limit is taken as the dimensions of  $\Delta A \to 0$ .



Figure 26: The point  $P = (x_i^*, y_j^*)$  in the rectangle  $R_{ij}$ .

To approximate the total mass of the lamina, we partition D into small rectangles (say  $R_{ij}$ ) and choose a point  $(x_i^*, y_j^*)$  inside  $R_{ij}$ . The mass of the lamina inside  $R_{ij}$  is approximately  $\rho(x_i^*, y_j^*)\Delta A_{ij}$ , where  $\Delta A_{ij}$  is the area of  $R_{ij}$ . Adding all such masses, we have the approximation

$$m \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_i^*, y_j^*) \Delta A_{ij}$$

If we then take the limit as  $m, n \to 0$ , we obtain

$$m = \iint_D \rho(x, y) dA$$

In a similar way, we can determine the moment of the lamina about the x-axis to be cc

$$M_x = \iint_D y\rho(x,y)dA$$

and the moment of the lamina about the y-axis to be

$$M_y = \iint_D x\rho(x,y)dA.$$

The centre of mass is located at coordinates  $(\overline{x}, \overline{y})$ , where

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}.$$

29.1 Example: find the centre of mass of a triangular lamina with vertices (0,0), (1,0) and (0,2) with constant density  $\rho_0$ .

29.2 Example: find the centre of mass of a rectangle with vertices (0,0), (2,0), (2,1) and (0,1) with density  $\rho(x,y) = 6x + 12y$ .



Notes.

### **30** Introduction to triple integrals

By the end of this section, you should be able to answer the following questions:

- How do you evaluate a triple integral?
- How do you use a triple integral to find the mass of a solid object with known density?
- How do you change the order of integration in a triple integral?

We can extend the definition of a double integral to a triple integral

$$\iiint_R f(x, y, z) dV,$$

where R is a region in  $\mathbb{R}^3$  and dV is an element of volume.

If R is a region in  $\mathbb{R}^3$  specified by

$$r(x,y) \leq z \leq s(x,y)$$
  

$$p(x) \leq y \leq q(x)$$
  

$$a \leq x \leq b$$
(14)

then

$$\iiint_R f(x, y, z) dV$$
  
=  $\int_a^b \left\{ \int_{p(x)}^{q(x)} \left[ \int_{r(x,y)}^{s(x,y)} f(x, y, z) dz \right] dy \right\} dx.$ 

In two dimensions, there are 2 possible orders of integration. In three dimensions, there are 6.

**30.1** Find the mass of a rectangular block with dimensions  $0 \le x \le L, \ 0 \le y \le W$  and  $0 \le z \le H$  if the density is  $\rho = \rho_0 + \alpha xyz$ .

**30.2** Evaluate  $\iiint_R z \ dV$  over the region R bounded by the surfaces  $x = 0, \ y = 0, \ z = 0$  and x + y + z = 1.

## 30.3 Changing the order of integration

Express the integral  $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) dz dy dx$ , in the orders dz dx dy and dy dz dx.



Notes.
# 31 Cylindrical coordinates

By the end of this section, you should be able to answer the following questions:

- What is the relationship between rectangular coordinates and cylindrical coordinates?
- How do you transform a triple integral in rectangular coordinates into one in terms of cylindrical coordinates?
- What is the Jacobian of the transformation?

x

Sometimes it is useful to use cylindrical coordinates in order to simplify the integral. This involves the transformation



$$= r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{15}$$

We now aim to calculate a small element of volume of a cylindrical shell. This will then show how in a triple integral we can transform from rectangular coordinates to cylindrical coordinates by substituting the transformation (15) and by making the change

 $dx \ dy \ dz \longrightarrow r \ dr \ d\theta \ dz.$ 

Consider the following diagram.



The important result is that the triple integral in rectangular coordinates transforms as follows:

$$\iiint_R f(x, y, z) \ dx \ dy \ dz = \iiint_C f(r \cos \theta, r \sin \theta, z) \ r \ dr \ d\theta \ dz.$$

31.1 A simple example: Find the volume of a cylinder of radius R and height H. (Ans.  $\pi R^2 H$ ) 31.2 Find the mass of the solid defined by the region contained within the cylinder  $x^2 + y^2 = 1$  below the plane z = 4 and above the paraboloid  $z = 1 - x^2 - y^2$ . The density at any given point in the region is proportional to the distance from the axis of the cylinder.





Notes.

# 32 Spherical coordinates

By the end of this section, you should be able to answer the following questions:

- What is the relationship between rectangular coordinates and spherical coordinates?
- How do you transform a triple integral in rectangular coordinates into one in terms of spherical coordinates?
- What is the Jacobian of the transformation?

Sometimes it is useful to use spherical coordinates in order to simplify the integral. This involves the transformation

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi.$$
 (16)

In this case  $\theta$  is longitude,  $\phi$  is co-latitude, and r the distance from the origin.



We now aim to calculate a small element of volume of a spherical shell. This will then show how in a triple integral we can transform from rectangular coordinates to spherical coordinates by substituting the transformation (16) and by making the change

$$dx \ dy \ dz \longrightarrow r^2 \sin \phi \ dr \ d\theta \ d\phi.$$

Consider the following diagram.



The important result is that the triple integral in rectangular coordinates transforms as <u>follows</u>:

$$\iiint_R f(x, y, z) \, dx \, dy \, dz$$
  
= 
$$\iiint_S f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) \, r^2 \sin \phi \, dr \, d\theta \, d\phi.$$

**32.1** A simple example: Find the volume of a sphere of radius R.

32.2 Find the mass of a sphere of radius R whose density is given by  $\rho(x, y, z) = e^{-(x^2+y^2+z^2)^{1/2}}$ .

32.3 Find the volume of the "ice cream cone" R between a sphere of radius a (centred at the origin) and the cone  $z = \sqrt{x^2 + y^2}$ .





Notes.

## **33** Moments of inertia (second moments)

By the end of this section, you should be able to answer the following questions:

- How do you locate the centre of mass of a solid object using a triple integral?
- How do you calculate the moments of inertia about the three coordinate axes?

The moment of inertia of a particle of mass m about an axis (x, y, or z) is defined to be  $mr^2$  where r is the distance from the particle to the axis.

It is sometimes referred to as rotational inertia and can be thought of as the rotational analogue of mass for linear motion. For example, linear kinetic energy can be expressed as  $\frac{1}{2}mv^2$ , and the rotational kinetic energy as  $\frac{1}{2}I\omega^2$ . Linear momentum is determined by the formula p = mv, while angular momentum is given by  $L = I\omega$ . In these examples, I is the moment of inertia and  $\omega$  the angular velocity.

As we have seen from previous examples, the mass of a solid with density  $\rho(x, y, z)$ occupying a region R in  $\mathbb{R}^3$  is given by

$$m = \iiint_R \rho(x, y, z) dV.$$

The moments about each of the three coordinate planes are

$$M_{yz} = \iiint_R x\rho(x, y, z)dV, \quad M_{xz} = \iiint_R y\rho(x, y, z)dV,$$
$$M_{xy} = \iiint_R z\rho(x, y, z)dV$$

The centre of mass is then located at the point  $(\overline{x}, \overline{y}, \overline{z})$  where

$$\overline{x} = \frac{M_{yz}}{m}, \ \overline{y} = \frac{M_{xz}}{m}, \ \overline{z} = \frac{M_{xy}}{m}.$$

The moments of inertia about each of the three coordinate axes work out to be

$$I_x = \iiint_R (y^2 + z^2)\rho(x, y, z)dV,$$
$$I_y = \iiint_R (x^2 + z^2)\rho(x, y, z)dV,$$
$$I_z = \iiint_R (x^2 + y^2)\rho(x, y, z)dV.$$

**33.1** Derive the integral formula for  $I_x$ 

33.2 Example: locate the centre of mass of a solid hemisphere of radius *a* with density proportional to the distance from the centre of the base. Find its moment of inertia about the *z*-axis.

Notes.

# 34 Conservative vector fields

By the end of this section, you should be able to answer the following questions:

- What is meant by a conservative vector field and a corresponding potential function?
- Given a potential function, how do you determine the corresponding conservative vector field?
- Given a conservative vector field, how do you determine a corresponding potential function?

## 34.1 Vector fields

In what follows, the notation is always

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$$
 or  $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ .

A vector field in the x-y plane is a vector function of 2 variables

$$F(r) = F(x, y) = (F_1(x, y), F_2(x, y)) = F_1(x, y)i + F_2(x, y)j.$$

That is, associated to a point (x, y) is the vector F(r).

**34.1.1** Example: F(r) = (-y, x) = -yi + xj.



Similarly a vector field in 3-D is a vector function of 3 variables

$$F(r) = F(x, y, z)$$
  
=  $(F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$   
=  $F_1(x, y, z)i + F_2(x, y, z)j + F_3(x, y, z)k$ 

## 34.1.2 Example: Newtonian gravitational field

$$\begin{split} \boldsymbol{F}(\boldsymbol{r}) &= -\frac{mMG}{||\boldsymbol{r}||^3} \boldsymbol{r} = \boldsymbol{F}(x, y, z) \\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}} \boldsymbol{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}} \boldsymbol{j} \\ &+ \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \boldsymbol{k} \end{split}$$



## 34.2 Gradient of a scalar field, conservative vector fields

Recall for a differentiable scalar function f(x, y) in two dimensions, we define

$$\operatorname{grad} f = \frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j}.$$

For a differentiable scalar function f(x, y, z) in three dimensions, we define

grad 
$$f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

Alternatively we define the differential operator

$$abla = oldsymbol{i} rac{\partial}{\partial x} + oldsymbol{j} rac{\partial}{\partial y} + oldsymbol{k} rac{\partial}{\partial z}$$

so  $\operatorname{grad} f = \nabla f$ .

## **34.2.1** Example: find the gradient of $f(x, y, z) = x^2 y^3 z^4$ .

Note  $\nabla f$  is a vector. It's length and direction are independent of the choice of coordinates.  $\nabla f$  (evaluated at a given point P) is in the direction of maximum increase of f at P.

You may see the scalar function f referred to as a scalar field. If a vector field  $\boldsymbol{v}$  and a scalar field f are related by  $\boldsymbol{v} = \nabla f$ , we call f a potential function and  $\boldsymbol{v}$  a conservative vector field.

34.2.2 Verify that the Newtonian gravitational field is conservative with potential function  $f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$ .



Given a conservative vector field, how can we determine a corresponding potential function? The next example outlines this procedure.

34.2.3 The vector field  $F(x,y) = (3+2xy)i + (x^2-3y^2)j$  is conservative. Find a corresponding potential function.



Can we still determine a potential function when the conservative vector field is in three dimensions?

34.2.4 The vector field  $F(x, y, z) = y^2 i + (2xy + e^{3z})j + 3ye^{3z}k$  is conservative. Find a corresponding potential function.



Is there a way of determining whether or not a given vector field is conservative? To answer this question, we need to go back to the study of line integrals.

Notes.

# 35 The fundamental theorem for line integrals, path independence

By the end of this section, you should be able to answer the following questions:

- How do you evaluate line integrals?
- What is the fundamental theorem for line integrals and its consequences?
- What is a path independent line integral and what are its connections with conservative vector fields and line integrals over closed curves?

## 35.1 Line integrals in the plane

Recall the definite integral  $\int_{a}^{b} f(x) dx$  gives the net area above the *x*-axis and below the curve y = f(x). We can generalise this.

Consider the following problem: How do we calculate the area of the region between the curve C in the x-y plane and its image on the surface z = f(x, y)?



If the curve C can be parametrised by  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$  for  $a \le t \le b$ , then the area in question is given by the formula

area = 
$$\int_C f(x, y) \, dS = \int_a^b f(x(t), y(t)) || \mathbf{r}'(t) || \, dt$$
,

where dS is the infinitesimal element of arclength of C.

#### 35.2 Work done by a force, line integrals of vector fields

We can also consider integrating a vector field over a curve in the plane.

In the case  $\mathbf{F}$  is a field of force you should already have an idea (from MATH1052/MATH1072) how to determine the work done by  $\mathbf{F}$  in moving a particle along a curve C. First recall the cases of:

- 1. Constant F in 1D over a distance d, the work done is given by W = Fd.
- 2. Constant **F** in 2D over a straight line between points P and Q,  $W = \mathbf{F} \cdot \overrightarrow{PQ}$ .

We can use these straightforward cases to derive the more general expression for work done by a variable force F(x, y) over a piecewise continuous smooth curve C:

$$W = \int_C \boldsymbol{F}(x, y) \cdot \boldsymbol{T}(x, y) \, dS,$$

where T(x, y) is a unit tangent vector to C at a given point (x, y) on C.

#### 35.3 Evaluating line integrals of vector fields

To evaluate

$$\int_{C} \boldsymbol{F}(x,y) \cdot \boldsymbol{T}(x,y) \, dS,\tag{17}$$

the strategy is to parameterise C and express all quantities in the line integral in terms of this parameterisation. Namely,

- 1. Parameterise C by specifying a vector function  $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$  with  $t \in [a, b]$  that describes C.
- 2. Write F(x, y) restricted to C as F(r(t)) = F(x(t), y(t)).
- 3. Write  $\mathbf{T}(x,y) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$ , where  $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$  is a tangent vector to C.
- 4. Write dS = || r'(t) || dt.
- 5. Evaluate the line integral as a definite integral in terms of the parameter t:

$$\int_{C} \boldsymbol{F}(x,y) \cdot \boldsymbol{T}(x,y) \, dS = \int_{a}^{b} \left[ \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{\boldsymbol{r}'(t)}{||\boldsymbol{r}'(t)||} \right] ||\boldsymbol{r}'(t)|| dt$$
$$= \int_{a}^{b} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt.$$

#### 35.4 Common notation

Let *C* be a piecewise continuous smooth curve in the *x-y* plane connecting points *A* and *B*. Let  $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$  be a vector field. The line integral in expression (17) is often expressed as

or

$$\int_C oldsymbol{F}(oldsymbol{r}) \cdot oldsymbol{d}oldsymbol{r}$$
 $\int_C F_1(x,y) dx + F_2(x,y) dy,$ 

both of which are *parameter independent* ways of writing the line integral in (17). The expressions use the notation  $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ ,  $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$ . Introducing a parameterisation  $\mathbf{r}(t)$  for  $t \in [a, b]$  allows us to evaluate the line integral. The expressions above are useful notations to remind us how to evaluate the line integrals.

**35.4.1** Example: let A = (0, 1), B = (1, 2). Evaluate  $\int_C ((x^2 - y)dx + (y^2 + x)dy)$ along the curve C given by: (i) the straight line from A to B; (ii) the parabola  $y = x^2 + 1$  from A to B.

Note the line integrals in the previous example were *path dependent*. In other words, they have different values for different paths.

We will now investigate path independent line integrals.

## 35.5 Line integrals of conservative vector fields, path independence.

If F is a continuous vector field with domain D, we say the line integral  $\int_C F \cdot dr$  is path independent if

$$\int_{C_1} oldsymbol{F} \cdot oldsymbol{d} oldsymbol{r} = \int_{C_2} oldsymbol{F} \cdot oldsymbol{d} oldsymbol{r}$$

for any two paths  $C_1$  and  $C_2$  in D that have the same end points.

#### 35.5.1 The fundamental theorem for line integrals

If C is a smooth curve determined by  $\mathbf{r}(t)$  for  $t \in [a, b]$  and f(x, y) is differentiable with  $\nabla f$  being continuous on C, then

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)).$$

Pro<u>of</u>:

One consequence is that for conservative vector fields  $\nabla f$ , we have

$$\int_{C_1} 
abla f \cdot oldsymbol{d} oldsymbol{r} = \int_{C_2} 
abla f \cdot oldsymbol{d} oldsymbol{r}.$$

That is, the line integral of a conservative vector field is path independent.

It turns out, the converse is also true. Suppose  $\boldsymbol{F}$  is continuous on an open, connected region D. If  $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$  is path independent in D, then  $\boldsymbol{F}$  is conservative. <u>Proof:</u>

*Open* region: every point in the region is the centre of some disc lying entirely in the region (ie. an open region doesn't include the boundary points).

Connected region: Any two points in D can be joined by a path lying entirely in D.

Another interesting result is that if  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is path independent in some region D, then  $\oint_{C'} \mathbf{F} \cdot d\mathbf{r} = 0$  for *every* closed path C' in D. Here the symbol " $\oint$ " indicates the integral is over a *closed* curve.

Proof:

Perhaps it is not surprising that the converse is also true. That is, if  $\oint_{C'} \mathbf{F} \cdot d\mathbf{r} = 0$ for *every* closed path C' in some region D, then  $\int_{C} \mathbf{F} \cdot d\mathbf{r}$  is path independent in D. Proof:

We are looking at these results carefully because we ultimately want a simple way of checking whether or not a vector field is conservative. We are not quite there yet, but in the next section, we will arrive at a surprisingly simple test for a conservative vector field.

Note also that more details of these proofs (with slightly more mathematical rigour) can be found in Stewart, pages 1099 – 1103.

Notes.

## 36 Green's theorem and a test for conservative fields

By the end of this section, you should be able to answer the following questions:

- What is Green's theorem and under what conditions can it be applied?
- How do you apply Green's theorem?
- Given a vector field in two dimensions, how can we test whether or not it is conservative?

#### 36.1 The story so far

The following diagram summarises the relationships between conservative vector fields, path independent line integrals and closed line integrals we have seen so far.



#### 36.2 Clairaut's theorem and consequences

Suppose a function of two variables f is defined on a disc D that contains the point (a, b). If the functions  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y \partial x}$  are both continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$
Say we have a conservative vector field  $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j}$ . This means that there exists an f(x, y) such that

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}.$$

An immediate consequence of Clairaut's theorem is that

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F_2}{\partial x}.$$

In otherwords, we have the following:

If 
$$\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j}$$
 is a conservative vector field, then  

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$
Let's add this to our diagram:

Let's add this to our diagram:



If we can reverse the new arrow, then we would have the criterion that we need! That is, the condition

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

would be a test for a conservative vector field. To do this, we require one more piece of the puzzle. That is Green's theorem.

#### 36.3 Green's theorem

Let *D* be a region in the *x-y* plane bounded by a piecewise-smooth, simple closed curve *C*, which is traversed with *D* always on the left. Let  $F_1(x, y)$ ,  $F_2(x, y)$ ,  $\frac{\partial F_1}{\partial y}$  and  $\frac{\partial F_2}{\partial x}$  be continuous in *D*. Then

$$\iint_{D} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy = \oint_{C} (F_1 dx + F_2 dy).$$

This theorem relates a double integral to a line integral over a closed curve. For example, we can use Green's theorem to evaluate complicated line integrals by treating them as double integrals, or vice versa.

Regarding our discussion on conservative vector fields, we have the following *corollary* to Green's theorem:

If 
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$
, then  $\oint_C \boldsymbol{F} \cdot \boldsymbol{dr} = 0$ .

Note that  $\boldsymbol{F} = F_1 \boldsymbol{i} + F_2 \boldsymbol{j}$ .

If we add this to our diagram, we can now link any four statements via the arrows. In otherwords all four statements are equivalent.



In particular, we now have a test to determine whether or not a given two dimensional vector field is conservative:

The vector field 
$$\boldsymbol{F}$$
 is conservative if and only if  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ .

36.3.1 Find the work done by the force  $F = x^2yi + xy^2j$  anticlockwise around the circle with centre at the origin and radius *a*.





**36.3.2** Evaluate the line integral  $\int_C 2xy \ dx + (x^2 + 3y^2) \ dy$ , where C is the path from (0,1) to (1,0) along  $y = (x-1)^2$  and then from (1,0) to (2,1) along y = x - 1.



**36.3.3** Evaluate  $\int_C (3+2xy)dx + (x^2-3y^2)dy$  where *C* is the curve parametrised by  $\mathbf{r}(t) = (1-\cos(\pi t))\mathbf{i} + (1+\sin^3(\pi t))\mathbf{j}$  for  $0 \le t \le 1/2$ . Notes.

## 37 On proving Green's theorem

Recall some important properties of double and line integrals.

$$\iint_{D_1 \cup D_2} f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA,$$
$$\iint_{D} (f_1(x, y) + f_2(x, y)) \, dA = \iint_{D} f_1(x, y) \, dA + \iint_{D} f_2(x, y) \, dA,$$

where  $D_1$  and  $D_2$  are disjoint (apart from sharing a boundary). If C consists of the segments  $C_1, \dots, C_4$  such that  $C = C_1 \cup C_2 \cup C_3 \cup C_4$ , then

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$$

With the understanding that

$$\int_C F_1 \, dx = \int_C (F_1 \, dx + 0 \, dy), \qquad \int_C F_2 \, dy = \int_C (0 \, dx + F_2 \, dy),$$

one can prove (Exercise!) that

$$\int_{C} (F_1 \, dx + F_2 \, dy) = \int_{C} F_1 \, dx + \int_{C} F_2 \, dy.$$

Moreover one can prove the following two results:

**1.** If C can be parametrized by  $\mathbf{r}(t) = t\mathbf{i} + g(t)\mathbf{j}$  with  $a \le t \le b$ , then

$$\int_C F_1(x,y) \ dx = \int_a^b F_1(x,g(x)) \ dx.$$

**2.** If C can be parametrized by  $\mathbf{r}(t) = h(t)\mathbf{i} + t\mathbf{j}$  with  $c \leq t \leq d$ , then

$$\int_C F_2(x,y) \ dy = \int_c^d F_2(h(y),y) \ dy.$$

We will use these results in the following.

**Green's Theorem.** Let *D* be a region in the *x-y* plane bounded by a piecewisesmooth, simple, closed, positively oriented curve *C*, and let  $F_1(x, y), F_2(x, y), \frac{\partial F_1(x,y)}{\partial y}, \frac{\partial F_2(x,y)}{\partial x}$  be continuous. Then,

$$\iint_{D} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_{C} (F_1 \, dx + F_2 \, dy).$$

In Sections 37.1-37.3, we prove the theorem for a particular class of regions. A general proof is beyond the scope of this workbook.

## 37.1 Region of type I

Recall that a region D is of type I if

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\},\$$

where  $g_1$  and  $g_2$  are continuous functions. We prove that for D of type I,

$$\iint_{D} \frac{\partial F_1}{\partial y} \, dA = -\oint_{C} F_1 \, dx.$$



# 37.2 Region of type II

A region D is of type II if

$$D = \{(x, y) \mid h_1(y) \le x \le h_2(y), \ c \le y \le d\},\$$

where  $h_1$  and  $h_2$  are continuous functions. Then similarly to the last subsection, by considering D as a type II region, we find that

$$\iint_{D} \frac{\partial F_2}{\partial x} \, dA = \oint_{C} F_2 \, dy.$$



#### 37.3 Region of both type I and II

If D is of *both* type I and II, we can combine the results of Sections 37.1 and 37.2, thereby obtaining Green's theorem:

$$\iint_{D} \left(\frac{\partial F_{2}}{\partial x} - \frac{\partial F_{1}}{\partial y}\right) dA = \iint_{D} \frac{\partial F_{2}}{\partial x} dA - \iint_{D} \frac{\partial F_{1}}{\partial y} dA = \oint_{C} F_{2} dy - \left(-\oint_{C} F_{1} dx\right)$$
$$= \oint_{C} (F_{1} dx + F_{2} dy).$$

#### 37.4 More general regions

If the planar region is neither type I nor type II, our strategy is to attempt to partition the region into subregions which are both type I and type II, then make use of the results of the previous calculations.

In the following two examples, partition the regions into subregions that are both type I and type II. Could we generalise the previous calculations to deal with subregions that are only either type I or II?





Notes.

# 38 Flux of a vector field

By the end of this section, you should be able to answer the following questions:

- What is the flux of a constant vector field across a flat surface in 3D?
- What is the flux of a vector field across a plane curve in 2D?

In this section we introduce the concept of *flux*: In three dimensions, the flux of a vector field across a given surface is defined to be the "flow rate" of the vector field through the surface.

Since many vector fields involve no motion (eg. electric fields, magnetic fields), this definition can be very difficult to comprehend at first. A nice context for working with flux in order to understand its definition is by considering the velocity vector of a fluid (so now we do have motion). In three dimensions, the flux of a fluid across a surface is given in units of volume per unit time. In other words, the flux tells us how much of the fluid (volume) passes through a given surface in one second.



Consider a river flowing at a constant velocity of 2m/s in only one direction. Now imagine placing a 3m square fishing net into the river so that it somehow stays perpendicular to the flow of the river. What is the flux of the water through the net?



Now if we rotate the net through an angle  $\theta$ , what is the flux through the net?

#### 38.1 Flux in 2D

Before we look at the flux of a vector field through more general surfaces, let's look at flux in two dimensions, by considering the flow of a two dimensional fluid through a curve in the x-y plane. Note that in this context of a fluid in 2D, flux has dimensions *area* per unit time.

To start, consider the problem of calculating the flux of a fluid with constant velocity v = 2i through a line segment C perpendicular to the flow, where C is given by

$$C = \{ (x, y) \mid x = 2, \ 2 \le y \le 6 \}.$$





Now consider calculating the flux of the velocity vector  $\boldsymbol{v}(x,y)$  in the x-y plane through a curve C.

We first divide C up into arcs of length  $\Delta S$ , and *approximate*  $\boldsymbol{v}$  as constant over each arc.



This constant vector over each arc shall be evaluated at a representative point in each arc, say  $P^* = (x^*, y^*)$ . We also approximate the arc as a straight line, so that

 $\Delta S \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx ||\mathbf{r}'(t)|| \ \Delta t.$ 



The component of  $\boldsymbol{v}$  which is perpendicular to C (over  $\Delta S$ ) is  $\approx \boldsymbol{v}(P^*) \cdot \boldsymbol{n}(P^*)$ . We then have

flux through one arc 
$$\approx \boldsymbol{v}(P^*) \cdot \boldsymbol{n}(P^*) \Delta S$$
.  
 $\Rightarrow$  total flux through  $C \approx \sum_i \boldsymbol{v}(P_i^*) \cdot \boldsymbol{n}(P_i^*) \Delta S_i$ .

If we take the limit as  $\Delta S \to 0$ , we obtain an exact expression for the flux over the entire curve C as a line integral:

Flux = 
$$\int_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS$$
,

where  $\boldsymbol{n}$  is a unit vector normal to C.

We use this expression as a *definition* of flux of any two dimensional vector field  $\boldsymbol{v}$  across a plane curve C. Note then that

dimensions of flux (in 2D) = (dimensions of  $\boldsymbol{v}$ ) × (distance).

#### 38.1.1 Evaluating flux in 2D

To evaluate the line integral in the definition of flux, we need a parametrisation of C, say  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$  for  $a \le t \le b$  (say). We define

$$\boldsymbol{r}'(t) = \dot{x} \,\, \boldsymbol{i} + \dot{y} \,\, \boldsymbol{j}.$$

A unit tangent vector to C is then given by

$$oldsymbol{T} = rac{oldsymbol{r}'(t)}{||oldsymbol{r}'(t)||}.$$

By the definition of vector cross product, and since k is a unit vector normal to the x-y plane, being careful of the direction of n, we can take

$$oldsymbol{n} = oldsymbol{T} imes oldsymbol{k} = rac{1}{||oldsymbol{r}'(t)||} \left|egin{array}{ccc} oldsymbol{i} & oldsymbol{j} & oldsymbol{k} \ \dot{x} & \dot{y} & 0 \ 0 & 0 & 1 \end{array}
ight|$$

$$= \frac{1}{||\boldsymbol{r}'(t)||} (\dot{y} \ \boldsymbol{i} - \dot{x} \ \boldsymbol{j})$$
  

$$\Rightarrow \boldsymbol{v} \cdot \boldsymbol{n} = \frac{\boldsymbol{v} \cdot (\dot{y} \ \boldsymbol{i} - \dot{x} \ \boldsymbol{j})}{||\boldsymbol{r}'(t)||}$$
  

$$= \frac{v_1 \dot{y} - v_2 \dot{x}}{||\boldsymbol{r}'(t)||},$$

where  $\boldsymbol{v}(x,y) = v_1(x,y)\boldsymbol{i} + v_2(x,y)\boldsymbol{j}$ . Noting also that in the integral we have  $dS = ||\boldsymbol{r}'(t)|| dt$ , we then have a means of evaluating the line integral (2D flux integral) as

$$\int_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS = \int_{t=a}^{t=b} (v_1(t)\dot{\boldsymbol{y}} - v_2(t)\dot{\boldsymbol{x}}) \, dt.$$



38.1.2 Calculate the flux of v = -yi + xj (in the positive x direction) across the line x = 2 (for  $2 \le y \le 6$ ).

## 38.2 Outward flux across a closed curve in the plane

Let C be a piecewise-smooth, simple closed curve. Let  $v_1(x, y)$ ,  $v_2(x, y)$  be continuous in the region bounded by C. (Note that these are some of the conditions of Green's theorem!)

The net outward flux of  $\boldsymbol{v} = v_1 \boldsymbol{i} + v_2 \boldsymbol{j}$  across C is given by

Net outward flux 
$$= \oint_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS$$

where  $\boldsymbol{n}$  is a unit vector normal to C, directed outward from the region bounded by C.

38.2.1 Calculate the outward flux of v = xyi + xyj across the curve from (2,0) to (-2,0) via the semicircle of radius 2 centred at the origin (for  $y \ge 0$ ) followed by the straight line from (-2,0) to (2,0).

Notes.

# **39** Divergence of a vector field (div)

By the end of this section, you should be able to answer the following questions:

- How do you calculate the divergence of a given vector field?
- What is the significance of divergence?
- How does it relate to flux?

In this section we introduce the concept of *divergence* of a vector field.

## **39.1** Calculating divergence

Let

$$\boldsymbol{v}(x,y,z) = v_1(x,y,z)\boldsymbol{i} + v_2(x,y,z)\boldsymbol{j} + v_3(x,y,z)\boldsymbol{k}$$

be a differentiable vector function. Then the function

div 
$$\boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \nabla \cdot \boldsymbol{v}$$

is called the divergence of  $\boldsymbol{v}$ . Note div  $\boldsymbol{v}$  is a scalar quantity.

Divergence has an analogous definition in two dimensions. For

$$F(x,y) = F_1(x,y)i + F_2(x,y)j \Rightarrow \operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

39.1.1 Example:  $v = xy^2i + xyzj + yz^2k$ . Find div v

#### **39.2** Understanding div in two dimensions.

Consider the flow of a two dimensional fluid with continuous velocity field  $\boldsymbol{v}(x,y) = v_1(x,y)\boldsymbol{i} + v_2(x,y)\boldsymbol{j}$ . Our aim is to calculate the outward flux from a small rectangle in the plane of area  $\Delta x \Delta y$  as in the diagram below.



We first approximate the flux across each of the four sides of the rectangle. In each case the approximation will be  $\boldsymbol{v} \cdot \boldsymbol{n}\Delta S$ , where we assume  $\boldsymbol{v}$  is constant over each edge. Also let  $x^* \in [x, x + \Delta x]$  and  $y^* \in [y, y + \Delta y]$  represent chosen points in each interval.

Edge 1: we evaluate  $\boldsymbol{v}$  at  $(x^*, y)$  and assume it is constant across the entire edge. An outwardly pointing unit normal vector is  $-\boldsymbol{j}$ .

flux 
$$\approx \boldsymbol{v}(x^*, y) \cdot (-\boldsymbol{j}) \Delta x$$
.

Edge 2: we evaluate  $\boldsymbol{v}$  at  $(x + \Delta x, y^*)$  and assume it is constant across the entire edge. An outwardly pointing unit normal vector is  $\boldsymbol{i}$ .

flux 
$$\approx \boldsymbol{v}(x + \Delta x, y^*) \cdot (\boldsymbol{i}) \Delta y.$$

Edge 3: we evaluate  $\boldsymbol{v}$  at  $(x^*, y + \Delta y)$  and assume it is constant across the entire edge. An outwardly pointing unit normal vector is  $\boldsymbol{j}$ .

flux 
$$\approx \boldsymbol{v}(x^*, y + \Delta y) \cdot (\boldsymbol{j}) \Delta x.$$

Edge 4: we evaluate  $\boldsymbol{v}$  at  $(x, y^*)$  and assume it is constant across the entire edge. An outwardly pointing unit normal vector is  $-\boldsymbol{i}$ .

flux 
$$\approx \boldsymbol{v}(x, y^*) \cdot (-\boldsymbol{i}) \Delta y$$
.

Combining all four terms gives an approximation to the net outward flux: net outward flux

$$\approx \left( \boldsymbol{v}(x + \Delta x, y^*) - \boldsymbol{v}(x, y^*) \right) \cdot \boldsymbol{i} \ \Delta y + \left( \boldsymbol{v}(x^*, y + \Delta y) - \boldsymbol{v}(x^*, y) \right) \cdot \boldsymbol{j} \ \Delta x$$

$$= \left( \frac{\boldsymbol{v}(x + \Delta x, y^*) - \boldsymbol{v}(x, y^*)}{\Delta x} \right) \cdot \boldsymbol{i} \Delta x \Delta y + \left( \frac{\boldsymbol{v}(x^*, y + \Delta y) - \boldsymbol{v}(x^*, y)}{\Delta y} \right) \cdot \boldsymbol{j} \Delta x \Delta y$$

$$= \left( \frac{v_1(x + \Delta x, y^*) - v_1(x, y^*)}{\Delta x} + \frac{v_2(x^*, y + \Delta y) - v_2(x^*, y)}{\Delta y} \right) \Delta x \Delta y$$

$$\approx \left( \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) \Delta x \Delta y.$$

$$= \operatorname{div}(\boldsymbol{v}) \Delta x \Delta y.$$

Hence, we have

$$\frac{\text{flux out of a rectangle}}{\text{area of rectangle}} \approx \text{div}(\boldsymbol{v}).$$

If we take the limit as the dimensions of the rectangle appproach 0, we have

$$\operatorname{div}(\boldsymbol{v}) = \lim_{\Delta A \to 0} \frac{\operatorname{flux} \text{ out of } \Delta A}{\Delta A}.$$

In other words,  $\operatorname{div}(v)$  is the "outward flux density" of v at a given point.

This concept generalises quite naturally to three dimensions:

$$\operatorname{div}(\boldsymbol{v}(x,y,z)) = \lim_{\Delta V \to 0} \frac{\operatorname{flux} \text{ out of } \Delta V}{\Delta V}.$$

In the context of fluids (our main focus so far) we can say  $\operatorname{div}(\boldsymbol{v}(x, y, z))$  measures the tendency of the fluid to "diverge" from the point (x, y, z).

## 39.3 Outward flux across a closed curve in the plane (revisited)

One final calculation uses the divergence to calculate the net outward flux of  $\boldsymbol{v}$  across a closed curve. We have already seen that we can evaluate this quantity by calculating  $\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS$ .

Now let D be a region in the x-y plane bounded by a piecewise-smooth, simple closed curve C, which is traversed with D always on the left. Let  $v_1(x, y)$ ,  $v_2(x, y)$  have continuous derivatives in D (again the conditions of Green's theorem!).



By the previous calculation involving divergence, we can also approximate the outward flux from the region by dividing D up into small rectangles and approximating the net outward flux across each rectangle. We know that for one rectangle,

outward flux of one rectangle  $\approx \operatorname{div}(\boldsymbol{v}(x^*, y^*))\Delta x\Delta y,$ 

where  $(x^*, y^*)$  is some point inside the rectangle. We repeat this for each rectangle containing part of the region D, so that

net outward flux across 
$$C \approx \sum \operatorname{div}(\boldsymbol{v}(x^*, y^*))\Delta x \Delta y$$
.

Taking the limit as  $\Delta x, \Delta y \to 0$ , we have

net outward flux across 
$$C = \iint_D \operatorname{div}(\boldsymbol{v}(x, y)) \, dA$$
,

the double integral of the region D.

To obtain the flux, we integrate the flux density over the region. Compare this with the context of mass density: to obtain the mass, we integrate the mass density over the region.

Finally, the two ways of calculating the same quantity must obviously be equal:

$$\oint_C \boldsymbol{v}(x,y) \cdot \boldsymbol{n} \ dS = \iint_D \operatorname{div}(\boldsymbol{v}(x,y)) \ dA.$$

## 39.4 Relationship to Green's theorem

We have seen how to evaluate the 2D flux integral:

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS = \int_{t=a}^{t=b} (v_1(t)\dot{\boldsymbol{y}} - v_2(t)\dot{\boldsymbol{x}}) \, dt.$$

This can be rewritten as

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS = \oint_C v_1 \, dy - v_2 \, dx.$$

If we define  $F_1(x,y) = -v_2(x,y)$  and  $F_2(x,y) = v_1(x,y)$ , we then have

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS = \oint_C F_1 \, dx + F_2 \, dy.$$

We also have

$$\operatorname{div}(\boldsymbol{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y},$$

so that

$$\iint_{D} \operatorname{div}(\boldsymbol{v}) \ dA = \iint_{D} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \ dA.$$

This tells us that in terms of the new vector field

$$\boldsymbol{F} = -v_2\boldsymbol{i} + v_1\boldsymbol{j} = F_1\boldsymbol{i} + F_2\boldsymbol{j},$$

the two ways of calculating flux are given by

$$\iint_{D} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy = \oint_{C} (F_1 dx + F_2 dy).$$

This is none other than Green's theorem. So the flux identity we obtained at the bottom of the previous page is just Green's theorem in disguise. We shall call this the *flux form* of Green's theorem.

39.4.1 Use the flux form of Green's theorem to calculate the outward flux of v = xyi + xyj across the curve from (2,0) to (-2,0) via the semicircle of radius 2 centred at the origin (for  $y \ge 0$ ) followed by the straight line from (-2,0) to (2,0).



39.4.2 For the following graphs of vector fields, determine whether the divergence is positive, negative or zero.







Notes.

# 40 Parametrisation of surfaces in $\mathbb{R}^3$

By the end of this section, you should be able to answer the following questions:

- What does it mean to parametrise a surface in  $\mathbb{R}^3$ ?
- How do you parametrise certain surfaces?

#### 40.1 Parametric surfaces

We have already seen two ways of representing a surface in  $\mathbb{R}^3$ : explicitly as z = f(x, y) or implicitly as F(x, y, z) = 0.

Another way of representing a surface S in  $\mathbb{R}^3$  is by a parametrisation. This is where the coordinate variables are functions of two parameters u and v:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

and the vector

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k}$$

traces out the surface as u, v vary over some region D in the "u-v plane". So for every point (u, v) in D, there corresponds a point on the surface S.

The following diagram shows the point P on the surface S which corresponds to the point (u, v) in the region D in the u-v plane. As (u, v) moves around all points in D, the point P moves around in S, tracing out the entire surface.



Note that a surface defined explicitly by z = f(x, y) is equivalent to a parametrisation

$$\boldsymbol{r}(x,y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x,y)\boldsymbol{k},$$

where we treat the coordinate variables x and y as the parameters. Note that we have not specified any bounds on the variables. Often the challenge is to not only find suitable functions for a parametrisation, but for a finite surface to determine bounds on the parameters.

# 40.2 Parametrising surfaces using cylindrical and spherical coordinates

We can use our knowledge of cylindrical and spherical coordinates to parametrise certain surfaces with which these coordinates are naturally associated.

Recall cylindrical coordinates:

$$x = r\cos\theta, \ y = r\sin\theta, \ z = z.$$

Setting exactly one of the cylindrical coordinates to a constant value necessarily gives a parametric surface.

Setting z = 2 with  $0 \le \theta \le 2\pi$ ,  $0 \le r \le 3$  describes a disc of radius 3, centred at the z axis lying in the plane z = 2:



Setting r = 5 with  $0 \le \theta \le 2\pi$ ,  $1 \le z \le 3$  describes the surface of a cylinder of radius 5 and of height 2 between z = 1 and z = 3:



Setting  $\theta = \pi/2$  with  $2 \le z \le 4$ ,  $0 \le r \le 1$  describes a rectangle lying in the *y*-*z* plane. Another description of the same surface would be x = 0,  $\{(y, z) \mid 0 \le y \le 1, 2 \le z \le 4\}$ :



**40.2.1** Parametrise the paraboloid  $z = 1 - x^2 - y^2$  for  $z \ge 0$ .
Recall spherical coordinates:  $x = r \cos \theta \sin \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \phi$ .

Setting exactly one of the spherical coordinates to a constant value necessarily gives a parametric surface.

Setting r = 2 with  $0 \le \theta \le 2\pi$ ,  $0 \le \phi \le \pi$  describes the surface of a sphere of radius 2 centred at the origin:



Setting  $\phi = \pi/3$  with  $0 \le r \le 2$ ,  $0 \le \theta \le 2\pi$  describes the open cone with angle  $\pi/3$  to the positive z-axis, the "mouth" of which lies on the sphere of radius 2 and with vertex located at the origin:



Setting  $\theta = 0$  with  $0 \le r \le 3, 0 \le \phi \le \pi$  describes the half disc of radius 3 lying in the *x-z* plane:



40.2.2 Parametrise the part of the sphere  $x^2 + y^2 + z^2 = 16$  that lies between the planes z = 2 and z = -2.



#### 40.3 Tangent planes

Let S be a surface parametrised by

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k}.$$

Here we find the tangent plane to S at a point P specified by r(a, b).

There are two important families of curves on S. One where u is a constant, the other where v is a constant. The diagram below shows the relationship between horizontal and vertical lines in D (in the u-v plane) and curves on S.



Setting u = a defines a curve on S parametrised by  $\mathbf{r}(a, v)$ , for all values of v such that (a, v) lies in D. A tangent vector to this curve at P is

$$oldsymbol{r}_v = rac{\partial x}{\partial v}(a,b)oldsymbol{i} + rac{\partial y}{\partial v}(a,b)oldsymbol{j} + rac{\partial z}{\partial v}(a,b)oldsymbol{k}.$$

Similarly setting v = b defines another curve on S parametrised by r(u, b). A tangent vector to this curve at P is

$$\boldsymbol{r}_u = \frac{\partial x}{\partial u}(a,b)\boldsymbol{i} + \frac{\partial y}{\partial u}(a,b)\boldsymbol{j} + \frac{\partial z}{\partial u}(a,b)\boldsymbol{k}.$$

If  $\mathbf{r}_u$  and  $\mathbf{r}_v$  are continuous and  $\mathbf{r}_u \times \mathbf{r}_v$  is never **0** inside D (we make an exception for points on the boundary of D), we call the surface *smooth* (it has no "kinks").

For a smooth surface,  $\mathbf{r}_u \times \mathbf{r}_v$  is a normal vector at any point inside D. This vector evaluated at (u, v) = (a, b) is also normal to the tangent plane at the point P = (x(a, b), y(a, b), z(a, b)).

The equation of the tangent plane at P is given by

$$(\boldsymbol{r}_u(a,b) \times \boldsymbol{r}_v(a,b)) \cdot ((x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) - \boldsymbol{r}(a,b)) = 0.$$

40.3.1 Find the tangent plane to the surface parametrised by  $r(u,v) = u^2 i + v^2 j + (u + 2v)k$  at the point (1, 1, 3).



Notes.

## 41 Surface integrals

By the end of this section, you should be able to answer the following questions:

- What is a surface integral?
- How do you calculate the area of a parametric surface?
- How do you use surface integrals in applications such as calculating the mass of a "surface lamina" and finding the average temperature over a surface.

#### 41.1 Area of a parametric surface

Let S be a smooth parametric surface given by

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k},$$

where we assume for simplicity that the parameter domain is a rectangle in the u-v plane. To calculate the area of S, we work through the following steps:

- 1. Partition S into small patches.
- 2. Approximate each patch by a parallelogram lying in the tangent plane to the corner of the patch closest to the u-v origin.
- 3. Calculate the area  $\Delta S$  of each parallelogram and add them to give an approximation to the area of S.
- 4. Take the limit as the dimensions of  $\Delta S \rightarrow 0$  to obtain an exact expression for the area.

Let's have a closer look at each step.

1. A partition of S into patches will correspond to a partition of D (in the u-v plane) into small rectangles.



The dimensions of the rectangles in D will be  $\Delta u \Delta v$ .

2. Let one of the edges of a single patch be defined from parameter values (u, v) to  $(u + \Delta u, v)$ .



Using Pythagoras' law in three dimensions, we can approximate the length of this edge as

where in this case we have used  $\Delta x = x(u + \Delta u, v) - x(u, v)$  etc (ie. the change is only in u). Similarly, for an edge of patch running from parameter values (u, v) to  $(u, v + \Delta v)$  the length of that edge will be approximately  $||\mathbf{r}_v||\Delta v$ .

At the corner of the patch corresponding to parameter values (u, v), we can define the two vectors  $\mathbf{r}_u \Delta u$  and  $\mathbf{r}_v \Delta v$  which form two sides of a parallelogram, the side lengths of which coincide with our approximations to the lengths of the edges of the patch.



3. The vector  $(\mathbf{r}_u \ \Delta u) \times (\mathbf{r}_v \ \Delta v)$  is normal to the surface (and hence the tangent plane) at that point. Its magnitude gives the area of the parallelogram we use to approximate the area of the patch  $\Delta S$ . We then have

$$\Delta S \approx ||\boldsymbol{r}_u \times \boldsymbol{r}_v|| \ \Delta u \ \Delta v.$$

Adding these approximations for each patch in S gives us an approximation to the area of S:

area of 
$$S \approx \sum_{i} \Delta S_{i} = \sum_{i} ||\boldsymbol{r}_{u_{i}} \times \boldsymbol{r}_{v_{i}}|| \Delta u_{i} \Delta v_{i}.$$

4. Finally taking the limit as  $\Delta u, \Delta v \to 0$  we obtain

surface area = 
$$\iint_{S} dS = \iint_{D} ||\boldsymbol{r}_{u} \times \boldsymbol{r}_{v}|| du dv.$$

41.1.1 Application: find the surface area of the paraboloid  $z = 1 - x^2 - y^2$ for  $z \ge 0$ .

#### 41.2 More on calculating surface integrals, applications

Let f(x, y, z) be a scalar function in  $\mathbb{R}^3$ . We can define the surface integral of f over a smooth parametric surface S in  $\mathbb{R}^3$  as

$$\iint_{S} f(x, y, z) \ dS = \iint_{D} f(\boldsymbol{r}(u, v)) || \boldsymbol{r}_{u} \times \boldsymbol{r}_{v} || \ du \ dv.$$

Surface integrals and double integrals have similar applications. Indeed, a double integral is merely a special case of a surface integral where the surface lies entirely in the x-y plane.

For example, if a thin sheet has the shape of a surface S and the mass density at the point (x, y, z) is  $\rho(x, y, z)$ , then the mass of the sheet is given by a surface integral:

mass of sheet 
$$= \iint_{S} \rho(x, y, z) \, dS$$

Another application is in calculating the average value of a function over a surface. Let S be a smooth surface in  $\mathbb{R}^3$ . Then the average value of the function f(x, y, z) over that surface is given by

average value over surface 
$$= \frac{1}{\text{area of } S} \iint_{S} f(x, y, z) \ dS.$$

If the surface S is a closed surface, it is convention to write

$$\oint_{S} f(x, y, z) \ dS$$

to represent the surface integral.

If S is a finite union of smooth surfaces  $S_1, S_2, \ldots, S_n$  that intersect only at their boundaries, then

$$\iint_{S} f(x, y, z) \ dS = \iint_{S_1} f(x, y, z) \ dS + \iint_{S_2} f(x, y, z) \ dS + \dots + \iint_{S_n} f(x, y, z) \ dS.$$

Closed surfaces are often unions of smooth surfaces as demonstrated in the following example.

41.2.1 The function  $T(x, y, z) = x^2 + y^2 + z^2 + 4$  gives the temperature at any point (x, y, z) on the surface of a solid hemisphere of radius 1 centred at the origin, defined for  $z \ge 0$ . Find the average temperature over the surface.



Notes.

## 42 Variable transformations in double integrals

Recall that, under a change of variable x = x(u), a definite integral transforms as

$$\int_{a}^{b} f(x) \, dx = \int_{c}^{d} f(x(u)) \frac{dx}{du} \, du,$$

where a = x(c) and b = x(d). We have seen how this generalises to iterated integrals, when changing from cartesian coordinates to polar, cylindrical or spherical coordinates. Here, we discuss how integrals transform under more general coordinate changes. For simplicity, emphasis will be on coordinate changes in two dimensions and on the corresponding iterated (double) integrals.

#### 42.1 Two-variable change of coordinates

In two dimensions, a change of coordinates is conveniently described by a surjective transformation

$$T: S \to R, \qquad (u, v) \mapsto (x, y),$$

where

$$x = x(u, v),$$
  $y = y(u, v).$ 

We shall assume that T and its first-order partial derivatives are continuous on S. A key property of T is that it maps any boundary of the region S in the u-v plane to a boundary of R in the x-y plane. Such a transformation is particularly useful if we can restrict the coordinates u, v to take values on a rectangle. After possibly applying a second transformation between this rectangle and the **unit square** 

$$\{(u, v) \in \mathbb{R}^2 \mid 0 \le u, v \le 1\},\$$

this prompts us to focus on the case where S itself is the unit square.

If T(u, v) = (x, y), then the point (x, y) is called the *image* of the point (u, v). We extend this definition to regions.

42.2 Find the image of the region  $\{(u, v) \mid 0 \le u, v \le 1\}$  under the transformation  $x = u^2 - v^2$ , y = 2uv



## 42.3 Jacobian

We can view the change of variables as a parameterisation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j}, \qquad (u,v) \in S,$$

of the region R in the x-y plane. This can then be used to analyse how the change of variables affects a double integral over R. To this end, we recall that a double integral over R arises as the limit of a sum over 'larger and larger' families of 'smaller and smaller' patches making up R:

$$\sum_{\text{patches}} f(x^*, y^*) \, \Delta S^* \ \to \ \iint_R f(x, y) \, dS.$$

Here,  $\Delta S^*$  is the area of the patch containing the point  $(x^*, y^*)$ .

By working out an approximation of the patch area  $\Delta S^*$ , and expressing it in terms of the u, v variables, we thus arrive at the formula

$$\iint_{R} f(x,y) \, dx \, dy = \iint_{S} f(x(u,v), y(u,v)) \Big| \frac{\partial(x,y)}{\partial(u,v)} \Big| \, du \, dv,$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$$

is called the **Jacobian** of the transformation T.

Question: Can one always find a variable transformation that maps from a rectangular or a unit square?

Consider a type I region R,

$$R = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

Set 
$$x(u, v) = (1 - u)a + ub,$$
  
 $y = (1 - v)g_1(x) + vg_2(x),$   
 $\Rightarrow y(u, v) = (1 - v)g_1((1 - u)a + ub) + vg_2((1 - u)a + ub).$   
 $0 \le u \le 1$   
 $0 \le v \le 1$ 

What is the Jacobian then? Because  $\frac{\partial x}{\partial v} = 0$ , the corresponding Jacobian is simply given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} b-a & 0 \\ \star & g_2((1-u)a+ub) - g_1((1-u)a+ub) \end{vmatrix}$$
$$= (b-a) \left[ g_2 \left( a + (b-a)u \right) - g_1 \left( a + (b-a)u \right) \right]$$

Of course, a similar change of variables applies if R is of type II.

**42.4** Example:  $R = \{(x, y) \mid 0 \le x \le 1, x^2 \le y \le x\}$ 



Notes.

## 43 Flux integrals and Gauss' divergence theorem

By the end of this section, you should be able to answer the following questions:

- What is a flux integral?
- How do you use a flux integral to calculate the flux of a vector field across a surface?
- What is Gauss' divergence theorem and under what conditions can it be applied?
- How do you apply Gauss' divergence theorem?

We have already been introduced to the idea of flux of a variable vector field across a curve (in  $\mathbb{R}^2$ ) and the flux of a constant vector field across rectangular surfaces (in  $\mathbb{R}^3$ ). In this section we look at calculating the flux across smoothly parametric surfaces.

#### 43.1 Orientable surfaces

Let S be a smooth surface. If we can choose a unit vector that is normal to S at every point so that n varies continuously over S, we call S an *orientable* surface. The choice of n provides S with an *orientation*. There are only ever two possible orientations.

An example of an orientable surface is the surface of a sphere. The two possible orientations are out of the sphere or into the sphere.

An example of a non-orientable surface is a Möbius strip (see Stewart page 1139).

The orientation of a surface is important when considering flux through that surface. The orientation we choose is always the direction of positive flux.

#### 43.2 The flux integral

For a vector field  $\boldsymbol{v}(x, y, z)$ , we are interested in the flux of  $\boldsymbol{v}$  across a smooth orientable parametric surface S in  $\mathbb{R}^3$ , parametrised by  $\boldsymbol{r}(u, v)$ , with u and v defined over some domain D. Let  $\boldsymbol{n}(u, v)$  be a unit vector normal to the surface S which defines the orientation of the surface (and hence the direction of positive flux).

It would be most convenient to consider the context of fluid flow with  $\boldsymbol{v}(x, y, z)$  being the velocity of a fluid at the point (x, y, z).

To calculate the flux through S, we work through the following steps:

- 1. Partition S into small patches.
- 2. Approximate each patch by a parallelogram lying in the tangent plane to the corner of the patch closest to the u-v origin.
- 3. Approximate the flux through each parallelogram of approximate area  $\Delta S$  and add them to give an approximation to the total flux through S.
- 4. Take the limit as the dimensions of  $\Delta S \rightarrow 0$  to obtain an exact expression for the flux.

Let's have a closer look at these steps.

- 1,2. Steps 1 and 2 are exactly the same as steps 1 and 2 on page 295-296 of our calculation of surface area.
  - 3. We approximate the flux through one patch by treating v as constant over the patch (ie. the patch is small enough for this to be a decent approximation). Since we have already approximated the shape of the patch as a parallelogram, we need to work out the flux of a constant vector through a parallelogram.

To this end, consider the parallelogram defined by the two (non-parallel) vectors  $\boldsymbol{a}$  and  $\boldsymbol{b}$ . If we take the area of the patch to be  $\Delta S$ , it can be seen from the diagram below that the flux (volume per unit time if  $\boldsymbol{v}$  is velocity) passing through the parallelogram is

flux across parallelogram  $\approx \boldsymbol{v} \cdot \boldsymbol{n} \Delta S$ .



We can take  $\boldsymbol{n} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}$ , and the area of the parallelogram is  $||\boldsymbol{a} \times \boldsymbol{b}|| \approx \Delta S$ . We then have

flux across parallelogram 
$$\approx \boldsymbol{v} \cdot \frac{\boldsymbol{a} \times \boldsymbol{b}}{||\boldsymbol{a} \times \boldsymbol{b}||} ||\boldsymbol{a} \times \boldsymbol{b}|| = \boldsymbol{v} \cdot (\boldsymbol{a} \times \boldsymbol{b}).$$

As shown previously, a patch of surface can be approximated by a parallelogram determined by the two vectors  $\mathbf{r}_u \Delta u$  and  $\mathbf{r}_v \Delta v$ . Hence we have

flux across one patch  $\approx \boldsymbol{v} \cdot \boldsymbol{n} \Delta S = \boldsymbol{v} \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v) \Delta u \Delta v.$ 

Note that we need to check that the vector  $\mathbf{r}_u \times \mathbf{r}_v$  points in the direction of positive flux. If not, we use  $\mathbf{r}_v \times \mathbf{r}_u$ .

Adding these approximations over the entire surface S, we obtain

flux across 
$$S \approx \sum_{i} \boldsymbol{v}_{i} \cdot \boldsymbol{n}_{i} \Delta S_{i} = \sum_{i} \boldsymbol{v}(u_{i}, v_{i}) \cdot (\boldsymbol{r}_{u_{i}} \times \boldsymbol{r}_{v_{i}}) \Delta u_{i} \Delta v_{i}.$$

4. To obtain an exact expression for the flux across S we take the limit as  $\Delta u, \Delta v \rightarrow 0$ .

flux across 
$$S = \iint_{S} \boldsymbol{v} \cdot \boldsymbol{n} \, dS = \iint_{D} \boldsymbol{v} \cdot (\boldsymbol{r}_{u} \times \boldsymbol{r}_{v}) \, du \, dv.$$

This expression is called a *flux integral* and is used to calculate the flux of any vector field across a smooth orientable surface, not just fluids with a given velocity field.

43.2.1 Calculate the net outward flux of F(x, y, z) = zi + yj + xk across the surface of the cylindrical solid given by  $\{(x, y, z) \mid x^2 + y^2 \le 1, 0 \le z \le 2\}$ .

#### 43.3 Gauss' divergence theorem

On page 279 we saw the flux form of Green's theorem:

$$\oint_C \boldsymbol{v}(x,y) \cdot \boldsymbol{n} \ dS = \iint_D \operatorname{div}(\boldsymbol{v}(x,y)) \ dA.$$

The left hand side is essentially a flux integral in two dimensions, with n being an outwardly pointing unit normal vector to the curve C. The right hand side was derived from our realisation of the divergence as the "flux density".

It would be natural to ask if it is possible to extend this result to three dimensions.

Given a vector field in three dimensions, F(x, y, z), we have seen that the net outward flux across a closed, smooth, orientable surface S is given by  $\oiint F \cdot n \, dS$ ,

where  $\boldsymbol{n}$  is an outwardly pointing unit normal.

We have also seen that its divergence  $(\operatorname{div} F)$  can be viewed as the flux density, so

$$\operatorname{div} \boldsymbol{F} = \lim_{\Delta V \to 0} \frac{\operatorname{flux} \text{ of } \boldsymbol{F} \text{ out of } \Delta V}{\Delta V}.$$

Hence we expect to be able to calculate the net outward flux across a closed, smooth, orientable surface S as the triple integral of the flux density (ie. divF) over the volume enclosed by S.

Indeed, this is true, with F and S subject to certain conditions. The result is known as *Gauss' divergence theorem*:

Let S be a piecewise smooth, orientable, closed surface enclosing a region V in  $\mathbb{R}^3$ . Let  $\mathbf{F}(x, y, z)$  be a vector field whose component functions are continuous and have continuous partial derivatives in V. Then

$$\iint_{S} \boldsymbol{F} \cdot \boldsymbol{n} \ dS = \iiint_{V} \operatorname{div}(\boldsymbol{F}) dV,$$

where  $\boldsymbol{n}$  is the outwardly directed unit normal to S.

This theorem connects the flux of a vector field out of a volume with the flux through its surface. It says that we can calculate the net outward flux either as a closed surface integral, or as a triple integral. 43.3.1 Use Gauss' divergence theorem to calculate the net outward flux of F(x, y, z) = zi + yj + xk across the surface of the cylindrical solid given by  $\{(x, y, z) \mid x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$ .

Notes.

## 44 On proving Gauss' divergence theorem

Recall triple integrals: 6 possible orders of integration, categorise into 3 depending on the inner *variable* i.e.

$$\iint_{D_{xy}} \left( \int_{g(x,y)}^{h(x,y)} f(x,y,z) dz \right) dA_{xy}$$
$$\iint_{D_{xz}} \left( \int_{g(x,z)}^{h(x,z)} f(x,y,z) dy \right) dA_{xz}$$
$$\iint_{D_{yz}} \left( \int_{g(y,z)}^{h(y,z)} f(x,y,z) dx \right) dA_{yz}$$

## 44.1 Special case proof

1.

2.

3.

Recall the integral identity stated in Gauss' Divergence Theorem:

$$\iiint_R \nabla \cdot \mathbf{F} dV = \oiint_{\partial R} \mathbf{F} \cdot \mathbf{n} \, dS,$$

where  $\partial R$  denotes boundary of R and **n** is the outward unit normal vector of  $\partial R$ .

Note: Writing  $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$ , we have

Here we prove the theorem in the case R can be represented as either a type 1, 2, or 3 region in  $\mathbb{R}^3$ . This is a special case proof on building block regions in  $\mathbb{R}^3$ . See Stewart (7 ed) page 1153.

We intend to equate the following terms

$$\iiint_{R} \frac{\partial F_{1}}{\partial x} dV = \bigoplus_{\partial R} F_{1} \mathbf{i} \cdot \mathbf{n} dS,$$
$$\iiint_{R} \frac{\partial F_{2}}{\partial y} dV = \bigoplus_{\partial R} F_{2} \mathbf{j} \cdot \mathbf{n} dS,$$
$$\iiint_{R} \frac{\partial F_{3}}{\partial z} dV = \bigoplus_{\partial R} F_{3} \mathbf{k} \cdot \mathbf{n} dS.$$

Notes.

## 45 Curl of a vector field

By the end of this section, you should be able to answer the following questions:

- How do you calculate the curl of a given vector field?
- What is the significance of curl?
- How do you test whether or not a given three dimensional vector field is conservative?

#### 45.1 Calculating curl

If (x, y, z) is a right handed Cartesian coordinate system and  $\boldsymbol{v}(x, y, z) = v_1 \boldsymbol{i} + v_2 \boldsymbol{j} + v_3 \boldsymbol{k}$ is a differentiable vector field, then define

$$\operatorname{curl}(\boldsymbol{v}) = 
abla imes \boldsymbol{v} = \left| egin{array}{cc} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \ rac{\partial}{\partial x} & rac{\partial}{\partial y} & rac{\partial}{\partial z} \ v_1 & v_2 & v_3 \end{array} 
ight|$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) \mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) \mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \mathbf{k}.$$

Note that  $\operatorname{curl}(\boldsymbol{v})$  is a vector field.

45.1.1 Example: let  $v = yz^2i + zx^2j + xy^2k$ . Find  $\operatorname{curl}(v)$ .

## 45.2 Understanding curl

For the rotation of a rigid body about a fixed axis with angular velocity  $\boldsymbol{w}$ , the velocity at a point P, whose position vector is  $\boldsymbol{r}$ , is given by  $\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r}$ .

If we choose the axis of rotation to be the z-axis, then  $\boldsymbol{w} = \omega \boldsymbol{k}$ . Calculate curl $(\boldsymbol{v})$ .

In general,  $\operatorname{curl}(\boldsymbol{v})$  characterises the rotation of a vector field. We will investigate this further in the next section.

### 45.3 Conservative fields revisited

It turns out that the curl of a vector field is exactly what we need to generalise the result at the bottom of page 254 to three dimensions.

Show that if F is a conservative vector field, then  $\operatorname{curl} F = 0$ .

Indeed, the diagram on page 254 that outlines our logic can be extended directly to the three dimensional case. The only difference is the condition which will serve as our test for conservative fields, namely  $\operatorname{curl} F = 0$ .

The proofs of the links in the diagram for the three dimensional case below are very similar to those used in the two dimensional case. The only detail that is significantly different is showing that if  $\operatorname{curl} \mathbf{F} = \mathbf{0}$  then  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ . Note also that  $\mathbf{F}$  must be a vector field defined everywhere in  $\mathbb{R}^3$  with continuous partial derivatives. The proof of that part of the diagram requires a generalisation of Green's theorem known as *Stokes' theorem*, which we will investigate in the next section.



The main consequence of this diagram is that we have the following test for a conservative vector field in three dimensions:

A vector field F is conservative if and only if curlF = 0.

# 45.3.1 Determine whether or not the vector field F = (1+yz)i+(1+xz)j+xyk is conservative.



Notes.

## 46 Stokes' theorem

By the end of this section, you should be able to answer the following questions:

- What is Stokes' theorem and under what conditions can it be applied?
- How do you apply Stokes' theorem?
- What is the circulation of a vector field?

## 46.1 Summary of surfaces and curves

Here we summarise the different types of curves and surfaces which we need to understand Stokes' theorem. Although most of these definitions have already been given, you may find it useful to have all of this information in one place so you can review at a glance.

#### 46.1.1 Surfaces

- *Smooth*: the surface normal vector depends continuously on the points on the surface.
- *Piecewise smooth*: the surface consists of finitely many smooth surfaces intersecting only at their boundaries.
- *Oriented* (or *orientable*): the direction of the positive normal vector can be continued uniquely and continuously across the whole surface (especially if the surface is piecewise smooth).

#### 46.1.2 Curves

- *Smooth*: the tangent at each point on the curve is unique and varies continuously.
- *Piecewise smooth*: the curve consists of finitely many smooth curves.
- *Simple*: the curve never intersects itself anywhere between its endpoints.
## 46.2 Stokes' theorem

Let S be a piecewise smooth, orientable surface in  $\mathbb{R}^3$  and let the boundary of S be a piecewise smooth, simple, closed curve C. Let  $\mathbf{F}(x, y, z)$  be a continuous vector function with continuous first partial derivatives in some domain containing S. Then

$$\iint_{S} (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n} \, dA = \oint_{C} \boldsymbol{F} \cdot \boldsymbol{dr},$$

where  $\boldsymbol{n}$  is a unit normal vector of S, and the integration around C is taken in the direction using the "right hand rule" with  $\boldsymbol{n}$ .

## 46.2.1 Relation to Green's theorem

Recall Green's theorem in the plane. It relates a line integral on a boundary to a double integral over a region in the plane. Roughly speaking, Stokes' theorem is a 3-D version of this: it relates a surface integral on a piece of surface (in 3-D) to a line integral on the boundary of the surface.

In fact, note that if the surface is in the *x-y* plane with  $\boldsymbol{n} = \boldsymbol{k}$ , Stokes' theorem reduces to Green's theorem, since the  $\boldsymbol{k}$  component of curl $\boldsymbol{F}$  is just  $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ .

46.2.2 Verify Stokes' theorem where C is the curve of intersection of the plane y+z=2 and the cylinder  $x^2+y^2=1$ , oriented counterclockwise when looking from above, and  $F = [-y^2, x, z^2]$ .

## 46.3 Further reading: Circulation

Let  $\boldsymbol{v}$  represent the velocity field of a fluid and C is a piecewise smooth, simple, closed curve. We have

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{dr} = \oint_C \boldsymbol{v} \cdot \boldsymbol{T} \ dS$$

where  $\boldsymbol{T}$  is a unit tangent vector in the direction of the orientation of the curve. The dot product  $\boldsymbol{v} \cdot \boldsymbol{T}$  is the component of  $\boldsymbol{v}$  in the direction of  $\boldsymbol{T}$  (and hence the curve), so we can interpret  $\oint_C \boldsymbol{v} \cdot \boldsymbol{T} \, dS$  as a measure of the tendency of the fluid to move around the curve C. We call this quantity the *circulation* of  $\boldsymbol{v}$  around C.

Now define a small circle  $C_a$  of radius *a* about a point  $P_0$ , such that the disc  $S_a$  enclosed by  $C_a$  is normal to the vector  $\boldsymbol{n}(P_0)$ . Our aim here is to better understand curl $\boldsymbol{v}$ .

Since curl $\boldsymbol{v}$  is continuous, we approximate curl $\boldsymbol{v}$  over  $S_a$  as curl $\boldsymbol{v}(P_0)$ . Stokes theorem then gives us

$$\begin{split} \oint_{C_a} \boldsymbol{v} \cdot \boldsymbol{dr} &= \iint_{S_a} \operatorname{curl} \boldsymbol{v} \cdot \boldsymbol{n} \ dS \\ &\approx \iint_{S_a} \operatorname{curl} \boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) \ dS \\ &= \operatorname{curl} \boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) \iint_{S_a} dS \\ &= \operatorname{curl} \boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) (\pi a^2) \\ \Rightarrow \operatorname{curl} \boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) &\approx \frac{1}{\pi a^2} \oint_{C_a} \boldsymbol{v} \cdot \boldsymbol{dr} \\ &\approx \frac{\operatorname{circulation around disc}}{\operatorname{area of disc}}. \end{split}$$

This approximation improves as  $a \to 0$ . Indeed

$$\operatorname{curl} \boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \oint_{C_a} \boldsymbol{v} \cdot \boldsymbol{dr}.$$

Note that this has a maximum value when  $\operatorname{curl} \boldsymbol{v}(P_0)$  and  $\boldsymbol{n}(P_0)$  have the same direction.

In particular, if we take  $\boldsymbol{n}(P_0)$  to be each of the coordinate unit vectors  $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ , we have the following: The  $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$  components of curl $\boldsymbol{v}(P_0)$  give the *circulation density* at  $P_0$  in planes normal to each of the  $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$  respectively. The magnitude of curl $\boldsymbol{v}(P_0)$  gives the maximum circulation density about  $P_0$  in a plane normal to curl $\boldsymbol{v}(P_0)$ .

## 46.4 Further reading: Curl fields and vector potentials

One immediate consequence is that if there are two different surfaces  $S_1$  and  $S_2$  satisfying the criteria of Stokes' theorem, both with the same boundary curve C, then

$$\iint_{S_1} \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_1 \ dS = \oint_C \boldsymbol{F} \cdot \boldsymbol{dr} = \iint_{S_2} \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_2 \ dS.$$

We have that if S is a closed surface satisfying all of the other criteria of Stokes' theorem, and if we define C to be any closed curve lying on S, so that  $S_1$  and  $S_2$  are two open surfaces whose union makes up S and whose common boundary is C, then

$$\oint_{S} \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n} \, dS = \iint_{S_{1}} \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_{1} \, dS + \iint_{S_{2}} \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_{2} \, dS$$

$$= \oint_{C} \boldsymbol{F} \cdot \boldsymbol{dr} + \oint_{-C} \boldsymbol{F} \cdot \boldsymbol{dr}$$

$$= \oint_{C} \boldsymbol{F} \cdot \boldsymbol{dr} - \oint_{C} \boldsymbol{F} \cdot \boldsymbol{dr} = 0,$$

since the orientation of C as a boundary to  $S_1$  will be in the opposite direction to that of  $S_2$ .

Let F be a vector field satisfying  $F = \operatorname{curl} G$  for some vector field G. We call F a curl field and G a corresponding vector potential.

The above result says that the net outward flux of a curl field across any closed surface is zero.

We can verify that  $\operatorname{div}(\operatorname{curl} \boldsymbol{G}) = 0$  for any vector field  $\boldsymbol{G}$ . Consequently we should not be too surprised by the above result, since Gauss' divergence theorem says that

$$\oint_{S} (\operatorname{curl} \boldsymbol{G}) \cdot \boldsymbol{n} \ dS = \iiint_{V} \operatorname{div}(\operatorname{curl} \boldsymbol{G}) \ dV = 0.$$

In fact, it turns out that we have the following test for curl fields:

Let F be a vector field whose components and their partial derivatives are continuous. If every closed surface in the domain of F only encloses points which are also in the domain of F, and if divF = 0, then there exists some G such that F = curlG. That is, F is a curl field. Notes.