1. a) \( \lim_{z \to 0} \frac{4z^3}{2^3 - 1337z^2} = \lim_{z \to 0} \frac{4(\frac{z}{2})^3}{(\frac{z}{2})^3 - 1337(\frac{z}{2})} = \lim_{z \to 0} \frac{4}{1 - 1337\left(\frac{z}{2}\right)^2} = 4 \)

b) \( \lim_{z \to 0} \frac{2^3}{2^3 + 1337z^2} = \infty \iff \lim_{z \to 0} \left[ \frac{\frac{2^3}{z^2}}{\frac{2^3}{z^2} + 1337} \right] = 0 \)

\( \iff \lim_{z \to 0} \left[ \frac{1}{2 + 1337z^2} \right] = 0 \iff \lim_{z \to 0} 2 + 1337z^2 = 0 \),

which is true.

c) \( \lim_{z \to 0} \frac{(az + b)^2}{(cz + d)^2} = \lim_{z \to 0} \frac{a(\frac{z}{2}) + b}{c(\frac{z}{2}) + d}^2 \)

\( \iff \lim_{z \to 0} \frac{(a + b)z}{(c + d)z} = \frac{a^2}{c^2} \)

2. For \( z = x + 0i, x \neq 0, \quad \frac{z}{z} = \frac{x}{x} = 1 \to \frac{a}{z} \to 0 \)

For \( z = 0 + 0i, y \neq 0, \quad \frac{z}{y} = \frac{0}{y} = -1 \to -1 \to y \to 0 \)

However, \( \lim \) must be independent of direction to approach; hence \( \lim \) does not exist.

3. a) Real and imaginary parts are defined on all of \( \mathbb{R}^2 \) (polynomial functions), so function is defined on \( \mathbb{C} \). Here \( u(x, y) = 2xy, v(x, y) = x^2 - y^2 \).

So \( u_x = 2y, u_y = 2x, v_x = 2x, v_y = 2y \).

(note: cf. ps. 5 in \( \mathbb{R}^2 \))
\[
\begin{align*}
C_{R_1} & \quad u_x = v_y \Rightarrow 2y = -2y \Rightarrow y = 0. \\
C_{R_2} & \quad u_y = v_x \Rightarrow 2x = -2x \Rightarrow x = 0.
\end{align*}
\]

Hence \( C_{R_1} \) only hold at \((0,0)\). Hence \( f^* \) is only differentiable at 0, so can't be analytic (not differentiable on any nbhd of any pt).

b) \( z = x - iy \), so
\[
\begin{align*}
\sin z &= \sin x \cosh(y) + i \cos x \sinh(y) \\
&= \sin x \cosh y - i \cos x \sinh y = u + iv
\end{align*}
\]
\( \Rightarrow u_x = \cos x \cosh y, \quad u_y = \sin x \sinh y, \quad v_x = \sin x \sinh y, \quad v_y = -\cos x \cosh y \)

\( C_{R_1}: u_x = v_y \iff \cos x = 0 \)

\( C_{R_2}: u_y = -v_x \iff \sin x = 0 \) or \( \sinh y = 0 \)

Hence \( C_{R_2} \) only hold at points of the form \((n\pi, 0)\). In particular, they don't hold on any nbhd of any pt. in \( C \Rightarrow \)
the \( f^* \) is nowhere analytic.

4. For this \( f^* \), we have \( u = x^4, \quad v = -(y-1)^4 \)
\( \Rightarrow u_x = 4x^3, \quad v_y = -4(y-1)^3, \quad u_y = v_x = 0 \).
So \( C_{R_1} \Rightarrow 4x^3 = -4(y-1)^3 \), which holds iff \( x^3 = -(y-1)^3 = (y-1)^3 \), i.e., \( x = y-1 \), i.e.
on the line \( y = x+1 \).
Note that $u$ & $v$ & partials are differentiable on $\Omega^2$.

Hence $f$ is differentiable precisely on the line $y = x + 1$ (C/R are necessary, so not differentiable off the line: sufficient conditions satisfied on the line).

(a) $f$ is not differentiable on any nodal of any ph, so not analytic anywhere.

5a) First, note that $u_r, u_\theta, v_r, v_\theta$ are defined & cts on the domain. Check C/R.

\[ u_r = v_\theta, \quad u_\theta = -v_r \]

Here $u = \Im f$, $v = \Re f$

\[ \Rightarrow u_r = i_r, \quad u_\theta = 0, \quad v_r = 0, \quad v_\theta = -1. \]

Hence (i) & (ii) hold.

Derivative is then $e^{-i\theta} (u_r + iv_r) = e^{-i\theta} \left( \frac{1}{r} \right) = \frac{1}{re^{i\theta}} - \frac{1}{2}$

(Note: Shows $\frac{d}{dz} \log z = \frac{1}{z}$ for this branch).

b) This is a rational $f^2$, so derivative exists on domain (note denominator doesn't vanish there).

Quotient rule \( \Rightarrow \)

\[ f'(z) = \frac{(z^3 + z)(4) - (4z+1)(3z^2 + 1)}{(z^3 + z)^2} = \frac{4z^3 + 4z - 12z^3 - 4z - 3z^2 - 1}{(z^3 + z)^2} = \frac{-8z^4 + 3z^2 + 1}{(2^3 + z)^2} \]

For both a) & b) $f^2$ is differentiable on whole domain, hence analytic on whole domain.